Another Look at the Describing Equations of Dynamics

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Abstract

We look at the fundamental equations of analytical dynamics from a different perspective. We discuss an additional way of evaluating the approaches for deriving the equations of motion, and we show that all of the fundamental equations can be viewed as projections of the force and moment balances onto directions affected by the velocity variables. We re-classify the existing approaches into two parts: Those based on vector variational principles, and those based on scalar variational principles. We discuss the relative merits and disadvantages of these approaches.

1 Introduction

During the 20th century, as the interest in complex dynamical systems has increased, researchers have looked into additional methods of describing their motion. Much of the research along these lines has been reported very subjectively. Further, as the need to deal with nonholonomic systems has increased, and the usefulness of *quasi-velocities* and methods that employ them has been recognized more and more, there has been a lot of debate on how best to make use of these variables.

The first use of quasi-coordinates when deriving equations of motion can be attributed to Heun, Hamel and Appell [1]. The resulting equations are called the *Gibbs-Appell* equations [2]. These equations are dated to the beginning of the 20th century, after developments in the late 19th century inspired research on handling nonholonomic systems. The drawback of the Gibbs-Appell equations is that a scalar function of accelerations needs to be calculated and then differentiated, which makes the approach cumbersome.

Another approach that makes use of quasi-velocities to derive equations of motion is *Kane's equations* [3], which refers to quasi-velocities as *generalized speeds*.[1] It has been shown in the literature that the Gibbs-Appell and Kane's equations are equivalent (e.g., [2]).

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[1]The term quasi-velocity precedes the term generalized speed by 50 years.

An issue of contention is whether Kane's equations were inspired from the Gibbs-Appell equations, or developed independently.

Another approach that makes use of quasi-velocities is *Lagrange's equations for quasi-coordinates* [4]. Here, one has the advantage of dealing only with velocity expressions. In addition, because Lagrange's equations are derivable from Hamilton's principle, Lagrange's equations are applicable to continuous (in space) systems. This property is valid whether generalized velocities or quasi-velocities are used.

The different approaches have their own advantages and disadvantages. On the other hand, certain arguments that were valid only a few years ago have become outdated. For example, Lagrange's equations deal with a lot of algebra, and they become cumbersome for large-order systems, which is why analytical techniques based on vector approaches began to enjoy more popularity in the second half of the 20th century. However, in the last few years there has been tremendous advances in symbolic manipulation software. We now have the capability of calculating complicated derivatives and equations of motion on personal computers (e.g., [5]).

In this paper, we discuss similarities and differences of the methods of analytical dynamics. We look at the physical interpretation of these equations. We provide a different classification of the describing equations. The goal in this exercise is to not be critical, but to provide additional perspective.

2 D'Alembert's Principle

Let us begin with a system of N particles. Particle i has mass m_i and its position, velocity and acceleration are described by the vectors $\mathbf{r}_i, \mathbf{v}_i$, and $\mathbf{a}_i$ $(i = 1, 2, ..., N)$, respectively. Denoting the first variation (hold time fixed, vary position) of the particle by $\delta\mathbf{r}_i$, the *Extended D'Alembert's principle* is written as

$$\sum_{i=1}^{N} m_i \mathbf{a}_i \cdot \delta\mathbf{r}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i \qquad i = 1, 2, ... \tag{1}$$

in which $\mathbf{F}_i$ denotes the sum of all impressed (or external) forces acting on the i-th particle. The forces that the particles exert on each other cancel.

Let the system under consideration have n degrees of freedom. It is then possible to express the motion of the i-th particle in terms as n *independent generalized coordinates* $q_1, q_2, ..., q_n$ as

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, ..., q_n, t) \tag{2}$$

One can express the first variation of $\mathbf{r}_i$ in terms of the generalized coordinates as

$$\delta\mathbf{r}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \tag{3}$$

where $\frac{\partial \mathbf{r}_i}{\partial q_k}$ are known as *kinematic variables* or *kinematic coefficients*. Let us investigate the properties of the kinematic coefficients. We note that they represent vectors, and if q_k is a position coordinate, the kinematic coefficient associated with it is dimensionless. In essence, the kinematic coefficient $\frac{\partial \mathbf{r}_i}{\partial q_k}$ represents the direction in which $\mathbf{r}_i$ is affected by q_k. The kinematic coefficients have the important property of

$$\frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_k} = \frac{\partial \mathbf{a}_i}{\partial \ddot{q}_k} \qquad k = 1, 2, ..., n \tag{4}$$

Hence, the kinematic coefficients can be calculated by using expressions for either one of position, velocity, or acceleration. The term to use is selected based on convenience.

To derive the equations of motion directly from D'Alembert's principle, we introduce Eq. (3) to Eq. (1), with the result

$$\sum_{k=1}^{n} \sum_{i=1}^{N} m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k = \sum_{k=1}^{n} \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \tag{5}$$

We denote the *generalized forces* by Q_k and define them as

$$Q_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \tag{6}$$

and we make use of the property that the variations of the generalized coordinates are independent themselves. Hence, Eqs. (5) can be written as n independent equations of motion in the form

$$\sum_{i=1}^{N} m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = Q_k \qquad k = 1, 2, ..., n \tag{7}$$

Let us analyze the form of the resulting equations. We began with the individual force balance equation for each particle ($m_i \mathbf{a}_i = \mathbf{F}_i$). Then, we took the dot product of these force balance equations with the kinematic coefficients and summed the resulting expressions over all the particles. Therefore, the k-th differential equation is the sum of the components of all force balances along the direction along the k-th kinematic coefficients. We have taken the individual force balances and projected them along the directions of the kinematic coefficients. Equations (6) - (7) are also known as *projection equations* [6].

Let us now extend the formulation above to rigid bodies. For simplicity, we initially consider a single rigid body. We denote the center of mass of the rigid body by G and write the position, velocity, and acceleration of a point on the body by

$$\mathbf{r} = \mathbf{r}_G + \boldsymbol{\rho} \qquad \mathbf{v} = \mathbf{v}_G + \boldsymbol{\omega} \times \boldsymbol{\rho} \qquad \mathbf{a} = \mathbf{a}_G + \boldsymbol{\alpha} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{8}$$

in which $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the angular velocity and angular acceleration of the body. By virtue of the definition of the center of mass $\int \boldsymbol{\rho} \, dm = \mathbf{0}$. To write the virtual displacement of a

point on the body, we recall that for general three dimensional motion angular velocity is not the derivative of a quantity, but a defined one. Hence, we denote the variation of an angular displacement by $\boldsymbol{\delta\theta}$ noting that the boldface extends over the entire term. We can write the variation of $\mathbf{r}$ directly from the velocity term, by replacing the time derivative with variations

$$\delta\mathbf{r} \;=\; \delta\mathbf{r}_G \;+\; \boldsymbol{\delta\theta} \times \boldsymbol{\rho} \tag{9}$$

Replacing m_i and $\mathbf{F}_i$ in Eq. (1) with dm and $d\mathbf{F}$, and the summation with integration over the body, we obtain

$$\int_{body} (\mathbf{a}_G + \boldsymbol{\alpha} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{\rho}) \cdot (\delta\mathbf{r}_G + \boldsymbol{\delta\theta} \times \boldsymbol{\rho}) dm = \int_{body} d\mathbf{F} \cdot (\delta\mathbf{r}_G + \boldsymbol{\delta\theta} \times \boldsymbol{\rho}) \tag{10}$$

After a few manipulations, we obtain the D'Alembert's principle for a rigid body as [2]

$$m\mathbf{a}_G \cdot \delta\mathbf{r}_G + \dot{\mathbf{H}}_G \cdot \boldsymbol{\delta\theta} = \mathbf{F} \cdot \delta\mathbf{r}_G + \mathbf{M}_G \cdot \boldsymbol{\delta\theta} \tag{11}$$

in which $\mathbf{H}_G$ is the angular momentum about the center of mass

$$\mathbf{H}_G = \int_{body} (\boldsymbol{\rho} \times \boldsymbol{\omega} \times \boldsymbol{\rho}) dm \tag{12}$$

with $\mathbf{F}$ and $\mathbf{M}_G$ denoting the resultant force and resultant moment about the center of mass.

For a system of N rigid bodies, D'Alembert's principle can be expressed as

$$\sum_{i=1}^{N} \Big(m_i \mathbf{a}_{Gi} \cdot \delta\mathbf{r}_{Gi} + \dot{\mathbf{H}}_{Gi} \cdot \boldsymbol{\delta\theta}_i \Big) = \sum_{i=1}^{N} \Big(\mathbf{F}_i \cdot \delta\mathbf{r}_{Gi} + \mathbf{M}_{Gi} \cdot \boldsymbol{\delta\theta}_i \Big) \tag{13}$$

where the notation is obvious.

As we did with particles, we can write the equations of motion for rigid bodies by direct use of D'Alembert's principle. Before we do this, we recall that the Newton-Euler formulation for a rigid body consists of the basic describing equations

$$\dot{\mathbf{p}} \;=\; m\mathbf{a}_G \;=\; \mathbf{F} \qquad \dot{\mathbf{H}}_G \;=\; \mathbf{M}_G \tag{14}$$

in which $\mathbf{p} = m\mathbf{v}_G$ is the linear momentum of the body. These equations were stated by Euler, in 1775, as the basic equations governing the motion of a rigid body.

It is convenient to write the variation of displacements in terms of generalized velocities as

$$\delta\mathbf{r}_{Gi} = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_{Gi}}{\partial q_k} \delta q_k = \sum_{k=1}^{n} \frac{\partial \mathbf{v}_{Gi}}{\partial \dot{q}_k} \delta q_k \qquad \boldsymbol{\delta\theta}_i = \sum_{k=1}^{n} \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{q}_k} \delta q_k \tag{15}$$

where we recognize $\frac{\partial \boldsymbol{\omega}_i}{\partial \dot{q}_k}$ as the kinematic coefficients associated with the angular velocity of the i-th body, $\boldsymbol{\omega}_i$. Introducing Eqs. (15) into Eq. (13) and considering the case when the generalized coordinates are independent, we obtain the equations of motion as

$$\sum_{i=1}^{N}\left(m_i\mathbf{a}_{Gi}\cdot\frac{\partial\mathbf{v}_{Gi}}{\partial\dot{q}_k}+\dot{\mathbf{H}}_{Gi}\cdot\frac{\partial\boldsymbol{\omega}_i}{\partial\dot{q}_k}\right)=Q_k \qquad k=1,2,...,n \tag{16}$$

in which Q_k $(k=1,2,...,n)$ are the generalized forces in the form

$$Q_k=\sum_{i=1}^{N}\left(\mathbf{F}_i\cdot\frac{\partial\mathbf{v}_{Gi}}{\partial\dot{q}_k}+\mathbf{M}_{Gi}\cdot\frac{\partial\boldsymbol{\omega}_i}{\partial\dot{q}_k}\right) \tag{17}$$

We can summarize the process involved in deriving the equations of motion directly by means of the D'Alembert's principle as

1) Obtain expressions for the change in the angular momentum and linear momentum, as well as resultant forces and moments;

2) Calculate the kinematic coefficients;

3) Take the force balance equation for the i-th mass ($m_i\mathbf{a}_{Gi}=\mathbf{F}_i$) and dot it to the kinematic coefficient associated with the translation of the center of mass, $(\frac{\partial\mathbf{v}_{Gi}}{\partial\dot{q}_k})$;

4) Take the moment balance equation for the i-th mass ($\dot{\mathbf{H}}_{Gi}=\mathbf{M}_{Gi}$) and dot it with the kinematic coefficient associated with the rotation of the body, $(\frac{\partial\boldsymbol{\omega}_i}{\partial\dot{q}_k})$;

5) Add the two expressions and sum over all the bodies.

Hence, we can view the equation of motion associated with the k-th generalized coordinate as the sum of the projections of the individual force and moment balances along the directions of the k-th kinematic coefficients. This observation is valid for D'Alembert's principle as well Lagrange's equations.

Let us now analyze this result more explicitly. Consider a rigid body and a set of unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ along a set of orthogonal coordinates attached to the body, so that $\boldsymbol{\omega}=\omega_1\mathbf{e}_1+\omega_2\mathbf{e}_2+\omega_3\mathbf{e}_3$. For example, using a 3-2-1 Euler angle transformation and the rotation angles ψ, θ, and ϕ, the angular velocity components along the body axes are

$$\omega_1=-\dot{\psi}\sin\theta+\dot{\phi} \qquad \omega_2=\dot{\psi}\cos\theta\sin\phi+\dot{\theta}\cos\phi \qquad \omega_3=\dot{\psi}\cos\theta\cos\phi-\dot{\theta}\sin\phi \tag{18}$$

and, if we begin with the unit vectors $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ and rotate them by ψ to get $\mathbf{a}_1'\mathbf{a}_2'\mathbf{a}_3'$, rotate the resulting set by θ to get $\mathbf{a}_1''\mathbf{a}_2''\mathbf{a}_3''$ and rotate this set to get by ϕ to get $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, we have

$$\frac{\partial\boldsymbol{\omega}}{\partial\dot{\psi}}=\mathbf{a}_3=\mathbf{a}_3' \qquad \frac{\partial\boldsymbol{\omega}}{\partial\dot{\theta}}=\mathbf{a}_2'=\mathbf{a}_2'' \qquad \frac{\partial\boldsymbol{\omega}}{\partial\dot{\phi}}=\mathbf{a}_1''=\mathbf{e}_1 \tag{19}$$

It follows from the above discussion that the equations of motion have the form

$$\text{for } \psi \qquad \dot{\mathbf{H}}_G\cdot\mathbf{a}_3=\mathbf{M}_G\cdot\mathbf{a}_3$$

$$\text{for } \theta \qquad \dot{\mathbf{H}}_G\cdot\mathbf{a}_2'=\mathbf{M}_G\cdot\mathbf{a}_2' \tag{20}$$

$$\text{for } \phi \qquad \dot{\mathbf{H}}_G\cdot\mathbf{e}_1=\mathbf{M}_G\cdot\mathbf{e}_1$$

This result can also be obtained independently of D'Alembert's principle, when one wants to develop a relationship between Lagrange's equations and Euler's equations [2].

Let us now compare the above equations with Euler's equations of motion for a rigid body

$$\dot{\mathbf{H}}_G \cdot \mathbf{e}_k \;=\; \mathbf{M}_G \cdot \mathbf{e}_k \qquad k = 1, 2, 3 \tag{21}$$

We cannot express the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as kinematic coefficients, as we cannot write them as partial derivatives of the angular velocity vector with respect to generalized velocities. However, we can express $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as $\mathbf{e}_k = \partial \boldsymbol{\omega} / \partial \omega_k$ $(k = 1, 2, 3)$.

The angular velocity components are not direct derivatives of rotational variables, but defined quantities. They are, in essence, quasi-velocities (generalized speeds) and are hence referred to as *nonholonomic*. Quasi-velocities are variables that are not necessarily the derivatives of the generalized coordinates, but combinations of the generalized velocities. This raises the question whether one can obtain the equations of motion directly from D'Alembert's principle in terms of quasi-velocities.

Consider an unconstrained system first. Denoting the quasi-velocities by $u_1, u_2, ..., u_n$, we express them as

$$u_k = \sum_{j=1}^{n} Y_{kj} \dot{q}_j + Z_k \qquad k = 1, 2, ..., n \tag{22}$$

where Y_{kj} and Z_k are functions of the generalized coordinates only. We can invert the above relationship as

$$\dot{q}_k = \sum_{j=1}^{n} W_{kj} u_j + X_k \qquad k = 1, 2, ..., n \tag{23}$$

in which W_{kj} and X_k are functions of the generalized coordinates only. It is clear that the square matrices $[Y]$ and $[Z]$, whose elements are Y_{kj} and Z_{kj} are the inverses of each other, and that this is the requirement for having a set of independent generalized speeds. Hence, a partial derivative (say, of $\mathbf{v}_G$ with respect to $\dot{q}_k$) can be expressed as

$$\frac{\partial \mathbf{r}_G}{\partial q_k} = \frac{\partial \mathbf{v}_G}{\partial \dot{q}_k} = \sum_{j=1}^{n} \frac{\partial \mathbf{v}_G}{\partial u_j} \frac{\partial u_j}{\partial \dot{q}_k} = \sum_{j=1}^{n} \frac{\partial \mathbf{v}_G}{\partial u_j} Y_{jk} \tag{24}$$

Similarly, we can express the inverse relationship as

$$\frac{\partial \mathbf{v}_G}{\partial u_k} = \sum_{j=1}^{n} \frac{\partial \mathbf{v}_G}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial u_k} = \sum_{j=1}^{n} \frac{\partial \mathbf{v}_G}{\partial \dot{q}_j} W_{jk} \tag{25}$$

The partial derivatives of $\mathbf{v}_G$ and $\boldsymbol{\omega}$ with respect to the generalized speeds u_k are called *partial velocities* [3], and they are defined as

$$\mathbf{v}_G^k = \frac{\partial \mathbf{v}_G}{\partial u_k} \qquad \boldsymbol{\omega}^k = \frac{\partial \omega}{\partial u_k} \tag{26}$$

Hence, similar to kinematic coefficients, we can ascribe a physical description to the partial velocities. The k-th partial velocity represents the direction that is affected by the k-th generalized speed. Note that the notation used here is slightly different than in [3] (subscript and superscript are reversed). We can express a time derivative (say, of $\mathbf{r}_G$) in terms of partial velocities as

$$\frac{d\mathbf{r}_G}{dt} = \mathbf{v}_G = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_G}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_G}{\partial t} = \sum_{k=1}^{n} \frac{\partial \mathbf{v}_G}{\partial \dot{q}_k} \dot{q}_k + \frac{\partial \mathbf{r}_G}{\partial t}$$

$$\mathbf{v}_G = \sum_{k=1}^{n} \mathbf{v}_G^k u_k + \mathbf{v}_G^t \tag{27}$$

where $\mathbf{v}_G^t$ is the time partial velocity. Let us now consider writing D'Alembert's principle in terms of partial velocities. We take Eqs. (16) - (17), multiply with W_{kj} and sum over j, which gives

$$\sum_{j=1}^{n} \sum_{i=1}^{N} \left(m_i \mathbf{a}_{Gi} \cdot \frac{\partial \mathbf{v}_{Gi}}{\partial \dot{q}_k} + \dot{\mathbf{H}}_{Gi} \cdot \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{q}_k} \right) W_{jk} = \sum_{j=1}^{n} \sum_{i=1}^{N} \left(\mathbf{F}_i \cdot \frac{\partial \mathbf{v}_{Gi}}{\partial \dot{q}_k} + \mathbf{M}_{Gi} \cdot \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{q}_k} \right) W_{jk} \tag{28}$$

Introducing Eq. (25) and its extension to angular velocities into the above equation, we obtain

$$\sum_{i=1}^{N} \left(m_i \mathbf{a}_{Gi} \cdot \mathbf{v}_{Gi}^k + \dot{\mathbf{H}}_{Gi} \cdot \boldsymbol{\omega}_i^k \right) = U_k \qquad k = 1, 2, ..., n \tag{29}$$

where U_k are the generalized forces associated with the generalized speeds in the form

$$U_k = \sum_{i=1}^{N} \left(\mathbf{F}_i \cdot \mathbf{v}_{Gi}^k + \mathbf{M}_{Gi} \cdot \boldsymbol{\omega}_i^k \right) \tag{30}$$

Equations (29)-(30) are the most general way of obtaining equations of motion by the direct application of D'Alembert's principle, as generalized velocities constitute a subset of generalized speeds. These equations are also known as *Kane's equations*, *projection equations*, or *fundamental equations* [2]. The formulation in Eq. (29) is slightly different than Kane's, in that the equations are expressed in terms of the linear and angular momenta. The advantage here is that the moment balance expressions, which are written above about the center of mass, can be replaced with moment balance equations about an arbitrary point, making the formulation more versatile.

The fundamental equations can be physically explained, the same way we explained Eqs. (16)-(17), as the sums of the projections of the force and moment balance equations along the directions of the k-th partial velocities. This, in essence, is the result obtained by Lesser [7], though through a different approach.

One advantage of using quasi-velocities is the additional freedom in selecting velocity variables. This is particularly evident when dealing with angular velocities. In many cases, the equations of motion can be considerably simplified by a judicious selection of the quasi-velocities. Another advantage is that for systems subjected to nonholonomic constraints one can find a set of independent generalized speeds and write the equations of motion directly in unconstrained form.

The fundamental equations can also be obtained from a scalar variational principle. This variational principle is the *Gauss' principle of least constraint* [2]. Related to the Gauss' principle are the Gibbs-Appell equations. To derive these equations, one begins with a quantity S, similar to kinetic energy, called by some *energy of acceleration*, having the form

$$S = \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{a}_i \cdot \mathbf{a}_i \tag{31}$$

and the resulting Gibbs-Appell equations of motion are

$$\frac{\partial S}{\partial \dot{u}_k} = U_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{v}_i^k \qquad k = 1, 2, ..., n \tag{32}$$

which can be shown to be the same as Kane's equations, both for particles as well as rigid bodies.

In between the D'Alembert's and Gauss' principles is the *Jourdain's Variational Principle*, which acts on the second variation (hold time and position fixed, vary velocity) [8]. For a system of particles, the principle has the form

$$\sum_{i=1}^{N} m_i \mathbf{a}_i \cdot \delta_1 \mathbf{v}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta_1 \mathbf{v}_i \tag{33}$$

and for rigid bodies the principle can be shown to be

$$\sum_{i=1}^{N} \Big(m_i \mathbf{a}_{Gi} \cdot \delta_1 \mathbf{v}_{Gi} + \dot{\mathbf{H}}_{Gi} \cdot \delta_1 \boldsymbol{\omega}_i \Big) = \sum_{i=1}^{N} \Big(\mathbf{F}_i \cdot \delta_1 \mathbf{v}_{Gi} + \mathbf{M}_{Gi} \cdot \delta_1 \boldsymbol{\omega}_i \Big) \tag{34}$$

Derivation of the fundamental equations from this principle is straightforward.

3   Quasi-Velocities and Lagrangian Mechanics

The previous section discussed how one can derive the equations of motion of a rigid body by direct use of D'Alembert's and Jourdain's principles. This is accomplished using either

generalized velocities or generalized speeds. We will refer to this way of obtaining the equations of motion as a *vector approach*.[2] Let us next analyze scalar approaches, namely Lagrange's equations.

Derivation of Lagrange's equations in terms of generalized coordinates and generalized velocities is well-known. In column vector format, and in terms of the kinetic and potential energies $\mathcal{T}$ and $\mathcal{V}$, we can write Lagrange's equations as

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \{\dot{q}\}} - \frac{\partial \mathcal{T}}{\partial \{q\}} + \frac{\partial \mathcal{V}}{\partial \{q\}} = \{Q\}^T \tag{35}$$

where $\{q\} = [q_1 \ q_2 \ ... \ q_n]^T$ and $\{Q\} = [Q_1 \ Q_2 \ ... \ Q_n]^T$. To express Lagrange's equations in terms of generalized speeds, we need to manipulate the terms in the above equation. We first express the kinetic energy in terms of $\{q\}$ and $\{u\}$ and denote the new expression by $\bar{\mathcal{T}}$. We then have

$$\frac{\partial \mathcal{T}}{\partial \{\dot{q}\}} = \frac{\partial \bar{\mathcal{T}}}{\partial \{u\}}\frac{\partial \{u\}}{\partial \{\dot{q}\}} = \frac{\partial \bar{\mathcal{T}}}{\partial \{u\}}[Y] \qquad \frac{\partial \mathcal{T}}{\partial \{q\}} = \frac{\partial \bar{\mathcal{T}}}{\partial \{q\}} + \frac{\partial \bar{\mathcal{T}}}{\partial \{u\}}\frac{\partial \{u\}}{\partial \{q\}} \tag{36}$$

We express the relationship between the generalized velocities and generalized speeds as

$$\{\dot{q}\} = [W]\{u\} + \{X\} \qquad \{u\} = [Y]\{\dot{q}\} + \{Z\} \tag{37}$$

and recognize that the generalized force vectors are related by

$$\{U\} = [W]^T\{Q\} \tag{38}$$

Introducing Eqs. (36) to Eq. (35), right multiplying the resulting equation by $[W]^T$ and considering Eq. (38) we write the Lagrange's equations in terms of quasi-velocities as

$$\frac{d}{dt}\frac{\partial \bar{\mathcal{T}}}{\partial \{u\}} + \frac{\partial \bar{\mathcal{T}}}{\partial \{u\}}[\mathcal{Z}] - \frac{\partial \bar{\mathcal{T}}}{\partial \{q\}}[W] = \{U\}^T \tag{39}$$

in which

$$[\mathcal{Z}] = \left(\frac{d}{dt}\frac{\partial \{u\}}{\partial \{\dot{q}\}} - \frac{\partial \{u\}}{\partial \{q\}}\right) \tag{40}$$

Hence, the equations of motion can be obtained starting from a scalar quantity, and they can be obtained in terms of the generalized coordinates or generalized speeds. We will refer to this way of obtaining the equations of motion as a *scalar approach*. Also included in this

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[2]Actually, one can consider Gauss' principle as a vector principle also, because it can be written as a third variation (hold time, position, and velocity fixed, vary acceleration).

category is, of course, the Gibbs-Appell equations. A variant of this Gibbs-Appell equations is the *Central Equation* [6].

Yet another combination of Lagrange's equations and generalized speeds is obtained when one makes use of Jourdain's variational principle. One can show that this principle can be written in scalar form as

$$\left(\frac{d}{dt} \frac{\partial \mathcal{T}}{\partial \{\dot{q}\}} - \frac{\partial \mathcal{T}}{\partial \{q\}} + \frac{\partial \mathcal{V}}{\partial \{q\}} - \{Q\}^T \right) \delta_1 \{\dot{q}\} \;=\; 0 \tag{41}$$

Now, expressing the variation of the generalized velocities as

$$\delta_1 \{\dot{q}\} \;=\; \frac{\partial \{\dot{q}\}}{\partial \{u\}} \delta_1 \{u\} \;=\; [W] \delta_1 \{u\} \tag{42}$$

Introducing the above equation into Eq. (41) and considering independent generalized speeds (whose Jourdain variations are also independent) we obtain

$$\left(\frac{d}{dt} \frac{\partial \mathcal{T}}{\partial \{\dot{q}\}} - \frac{\partial \mathcal{T}}{\partial \{q\}} + \frac{\partial \mathcal{V}}{\partial \{q\}} - \{Q\}^T \right) [W] \;=\; \{0\}^T \tag{43}$$

The above equation is known as *Maggi's equation* [6] and it is particularly useful when dealing with constrained systems. For a constrained system, $[W]$ is a rectangular matrix and one ends up with a set of independent equations. By contrast, Lagrange's equations for quasi-coordinates are for unconstrained systems and they require introduction of Lagrange multipliers for constrained systems.

One can classify the different approaches for deriving the equations of motion as: 1) Vector approaches based on the direct use of D'Alembert's or Jourdain's principles, and 2) Scalar approaches based on Hamilton's and Gauss' principles, such as Lagrange's equations or Gibbs-Appell equations. In both approaches, one can use either generalized velocities, or generalized speeds. Use of generalized speeds can make the derivation of the equations of motion simpler in many cases.

It then becomes a matter of choice to select the approach to use when writing the equations of motion. Each approach has advantages and disadvantages. For example, when using D'Alembert's principle one calculates accelerations and derivatives of the angular momentum, which involves a lot of algebra. By contrast, when using Lagrange's equations, one only deals with velocities and angular velocities. However, a time derivative is still taken, and because the approach requires partial derivatives, the algebraic burden can become overwhelming. These arguments have to be considered in light of the fact that there exist today powerful symbolic manipulation software packages and a comparison of the algebra involved in different approaches may be a moot point. Another issue that needs to be considered is the ease with which the resulting equations of motion lend themselves to numerical integration.

Table 1 summarizes the choices for deriving equations of motion using analytical techniques.

	Velocity Variables Used	
Type of Approach	Generalized velocities	Generalized speeds
Vector approaches (D'Alembert, Jourdain)	D'Alembert's principle Eqs. (16), (17)	Fundamental equations Eqs. (29), (30)
Scalar approaches (Hamilton, Gauss)	Lagrange's equations Eq. (35)	Gibbs-Appell equations, Eq. (32) Lagrange's eqs. for quasi-coordinates Eq. (39), Maggi (43)

Table 1. Summary of Analytical Techniques for Deriving Equations of Motion

An area where the direct use of D'Alembert's principle with generalized speeds has a distinct advantage over Lagrangian mechanics is with nonholonomic systems. When using D'Alembert's principle, one can select a set of independent generalized speeds and derive the equations of motion directly. This is not the case with Lagrangian mechanics, where one first needs to derive the equations of motion in terms of the Lagrange multipliers and then eliminate the Lagrange multipliers.

An area where Lagrangian mechanics has superiority over the direct use of D'Alembert's principle is in continuous systems, as in flexible bodies. Here, one makes use of Hamilton's principle and derives Lagrange's equations for continuous systems, in terms of partial differential equations [4]. This is not possible with the direct use of D'Alembert's principle, as this principle is not an integral principle.

4 Illustrative Examples

The first example is from vehicle dynamics and it illustrates the advantages associated with using a vector based approach. Consider Fig. 1, representative of a generic vehicle, driven by forces at C and D. The XY frame is inertial and the xy frame is attached to the vehicle, which moves according to the constraint

$$\mathbf{v}_A \cdot \mathbf{j} = 0 \tag{44}$$

so that the velocity of point A is always along the x direction. Using the generalized coordinates of X, Y and θ, where X and Y are the coordinates of the center of mass, we express the velocity of A as

$$\mathbf{v}_A = (\dot{X}\cos\theta + \dot{Y}\sin\theta)\mathbf{i} + (-\dot{X}\sin\theta + \dot{Y}\cos\theta - L\dot{\theta})\mathbf{j} \tag{45}$$

so that the constraint can be written as

$$-\dot{X}\sin\theta + \dot{Y}\cos\theta - L\dot{\theta} = 0 \tag{46}$$

We recognize this constraint to be nonholonomic. Let us first make use of the direct application of D'Alembert's principle and generalized speeds, which we select as $u_1 = \mathbf{v}_A \cdot \mathbf{j}, u_2 = \dot{\theta}$.

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We need to generate the partial velocities as well as the acceleration of the center of mass and rate of change of angular velocity. We have

$$\boldsymbol{\omega} = u_2\mathbf{k} \qquad \mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A} = u_1\mathbf{i} + Lu_2\mathbf{j}$$

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{C/A} = (u_1 - hu_2)\mathbf{i} \qquad \mathbf{v}_D = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{D/A} = (u_1 + hu_2)\mathbf{i} \qquad (47)$$

so that

$$\mathbf{v}_G^1 = \mathbf{i} \quad \mathbf{v}_G^2 = L\mathbf{j} \quad \mathbf{v}_C^1 = \mathbf{i} \quad \mathbf{v}_C^2 = -h\mathbf{i} \quad \mathbf{v}_D^1 = \mathbf{i} \quad \mathbf{v}_D^2 = h\mathbf{i} \quad \boldsymbol{\omega}^1 = \mathbf{0} \quad \boldsymbol{\omega}^2 = \mathbf{k} \qquad (48)$$

and the acceleration of G and the change in angular momentum are

$$\mathbf{a}_G = (\dot{u}_1 - Lu_2^2)\mathbf{i} + (L\dot{u}_2 + u_1u_2)\mathbf{j} \qquad \dot{\mathbf{H}}_G = I_G\dot{u}_2\mathbf{k} \qquad (49)$$

Noting that the applied forces are $\mathbf{F}_C = F_C\mathbf{i}$ and $\mathbf{F}_D = F_D\mathbf{i}$, the equations of motion become

$$m\mathbf{a}_G \cdot \mathbf{v}_G^1 + \dot{\mathbf{H}}_G \cdot \boldsymbol{\omega}^1 = \mathbf{F}_C \cdot \mathbf{v}_C^1 + \mathbf{F}_D \cdot \mathbf{v}_D^1 \quad \rightarrow \quad m(\dot{u}_1 - Lu_2^2) = F_C + F_D$$

$$m\mathbf{a}_G \cdot \mathbf{v}_G^2 + \dot{\mathbf{H}}_G \cdot \boldsymbol{\omega}^2 = \mathbf{F}_C \cdot \mathbf{v}_C^2 + \mathbf{F}_D \cdot \mathbf{v}_D^2 \quad \rightarrow \quad (I_G + mL^2)\dot{u}_2 + mLu_1u_2 = h(F_D - F_C) \quad (50)$$

We can interpret these equations as the force balance along the x direction and moment balance about point A. Next, let us solve this problem using scalar principles. The kinetic energy, in terms of the generalized velocities, has the form

$$\mathcal{T} = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}I_G\dot{\theta}^2 \qquad (51)$$

The potential energy is zero. Considering that the constraint is given by Eq. (46), the virtual work can be written as

$$\delta\hat{W} = \mathbf{F}_C \cdot \delta\mathbf{r}_C + \mathbf{F}_D \cdot \delta\mathbf{r}_D + \lambda\delta f$$

$$= (F_C + F_D)\cos\theta\delta X + (F_C + F_D)\sin\theta\delta Y + (F_D - F_C)h\delta\theta$$

$$+ \lambda(\delta X\sin\theta - \delta Y\cos\theta + L\delta\theta) \qquad (52)$$

where λ is the Lagrange multiplier. The equations of motion are obtained as

$$m\ddot{X} = (F_C + F_D)\cos\theta + \lambda\sin\theta$$

$$m\ddot{Y} = (F_C + F_D)\sin\theta - \lambda\cos\theta \qquad (53)$$

$$I_G\ddot{\theta} = (F_D - F_C)h + L\lambda$$

These equations can be manipulated to yield Eqs. (50), but it is clear that the effort involved is much larger than the direct use of D'Alembert's principle. This process can be simplified by using Maggi's equation. We relate the generalized coordinates and generalized speeds by

$$
\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_A \\ \dot{\theta} \end{bmatrix}
\tag{54}
$$

Combining Eqs. (43) and the coefficient matrix in the above equation, we obtain two independent equations in the form

$$
m\ddot{X}\cos\theta \,+\, m\ddot{Y}\sin\theta \,=\, F_C \,+\, F_D
$$

$$
m\ddot{X}\sin\theta \,+\, m\ddot{Y}\cos\theta \,+ I_G\ddot{\theta} \,=\, (F_C \,+\, F_D)h
\tag{55}
$$

and by a proper substitution of the quasi-velocities, these equations can be shown to be the same as the equations of motion derived earlier. Note that when using Maggi's equation the Lagrange multipliers need not be considered, which is an advantage of this approach over the constrained coordinate approach. However, one still has to convert from the generalized coordinates to the generalized speeds, once the independent equations of motion are obtained.

Next, let us use Lagrange's equations for quasi-coordinates. We select the same generalized speeds as before. This procedure cannot be carried out for the general case of constrained systems, as the effects of the constraints cannot be accounted for properly. The kinetic energy has the form

$$
\bar{T} = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\dot{\theta}^2 = \frac{1}{2}mu_1^2 + \frac{1}{2}(I_G + mL^2)u_2^2
\tag{56}
$$

We note that the kinetic energy is not a function of the generalized coordinates, so that $\partial\bar{T}/\partial q_k = 0$ ($k = 1, 2, 3$). The first term in Eq. (39) yields the first terms in Eq. (50). We now need to evaluate the second term in Eq. (39), namely $\frac{\partial\bar{T}}{\partial\{u\}}[\mathcal{Z}]$. To evaluate $[\mathcal{Z}]$, we first need to write $\{u\} = [Y]\{\dot{q}\}$. This expression can be written in many ways, as $\{u\} = [u_1 \; u_2]^T$ is a vector of order 2, while $\{q\} = [q_1 \; q_2 \; q_3]^T$ is of order 3. Recalling the definitions of the generalized speeds

$$
u_1 = v_A = (\dot{X}\cos\theta + \dot{Y}\sin\theta) \qquad u_2 = \dot{\theta}
\tag{57}
$$

so that if we write $[Y]$ as

$$
[Y] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
\tag{58}
$$

it becomes clear after a few steps that the correct equations of motion are not reached. The constraints were not properly accounted for. We need to write the expression for u_2 in such a way that we satisfy the relationship

$$
\frac{\partial\bar{T}}{\partial\{u\}}[Y] = \frac{\partial\bar{T}}{\partial\{\dot{q}\}}
\tag{59}
$$

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with the second row of $[Y]$ being unknowns. This approach eventually leads to the correct equations, but the procedure is cumbersome and it is not a valid approach for the general case. Hence, we conclude that for nonholonomic constrained systems, the direct application of D'Alembert's principle is more suitable than Lagrange's equations.

Next, let us consider a lightly flexible beam undergoing combined rigid and elastic motion. The beam is free at one end and is attached to a ball and socket joint at the other, as shown in Fig. 2. We will view the motion using the primary and secondary frame approaches [2]. We begin with an inertial XYZ frame, where the beam axis is the X axis and the flexibility is along the Y direction. We rotate the beam to its new position and the local body axes are now xyz with the x axis denoting the beam axis. The components of the angular velocity of the reference frame are given by Eq. (18). The deformation of a point on the beam is expressed by

$$\mathbf{r}(x,t) = x\mathbf{i} + v(x,t)\mathbf{j} \tag{60}$$

in which $v(x,t)$ is the elastic deformation. The velocity of this point is obtained by simple differentiation as

$$\mathbf{v}(x,t) = \dot{v}(x,t)\mathbf{j} + \boldsymbol{\omega} \times (x\mathbf{i} + v(x,t)\mathbf{j})$$

$$= -\omega_z v(x,t)\mathbf{i} + (x\omega_z + \dot{v}(x,t))\mathbf{j} + (\omega_x v(x,t) - x\omega_y)\mathbf{k} \tag{61}$$

A number issues are of interest: First, writing the velocity of a point using generalized velocities would be very cumbersome, so that using quasi-velocities is preferable. Second, the equations of motion cannot be obtained by direct use of D'Alembert's principle, unless the elastic motion is discretized beforehand. Finally, the above model contains substantial simplification, as it ignores the axial deformation as well as the shortening of the projection. A model that includes these effects is even more complicated. Hence, it is preferable to use a scalar integral principle, as well as quasi-velocities. This principle is, of course, the Extended Hamilton's principle. The kinetic energy has the form

$$\mathcal{T} = \int_0^L \mathbf{v}(x,t) \cdot \mathbf{v}(x,t)dm$$

$$= \int_0^L \left(\dot{v}^2 + v^2(\omega_x^2 + \omega_z^2) + x^2(\omega_y^2 + \omega_z^2) + 2x(\omega_z\dot{v} - \omega_x\omega_y v) \right) dm \tag{62}$$

By applying the Extended Hamilton's principle and Lagrange's equations for quasi-coordinates, one can obtain the equations of motion in hybrid form. We get three differential equations for the angular velocities, and a partial differential equation for the flexible motion. The procedure is outlined in [9] and [10].

In [6] a procedure is outlined, based on the direct use of D'Alembert's principle and the projection equations, to derive the equations of motion and boundary conditions of a deformable body in hybrid form. However, the procedure requires the use of potential energy, which implies that some use has been made of Hamilton's principle.

5　Conclusions

We look at the describing equations of analytical dynamics from a different perspective. We show that the equations of motion are basically the projection of the fundamental equations of force and moment balances along directions affected by the kinematic coefficients. This holds true for generalized velocities as well as for quasi-velocities. The equations of motion can be classified as either being based on vector principles, primarily direct use of D'Alembert's or Jourdain's principles, or on scalar principles, such as Hamilton's and Gauss'. Advantages and disadvantages of the various approaches are discussed.

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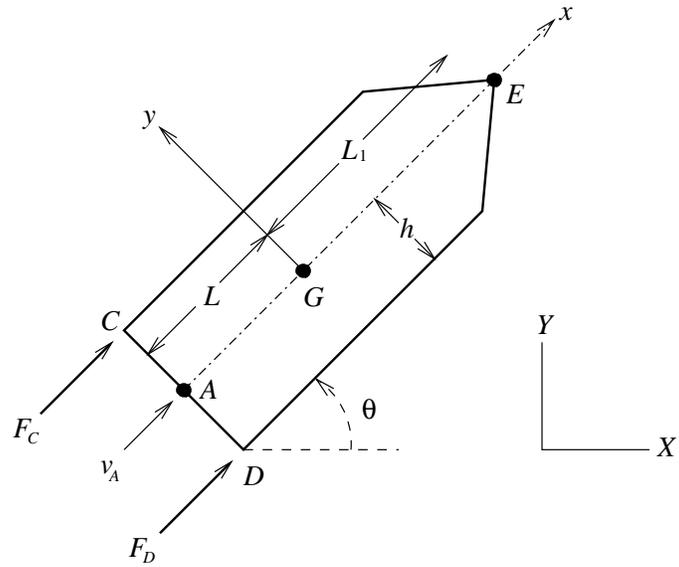
Figure 1. The vehicle.
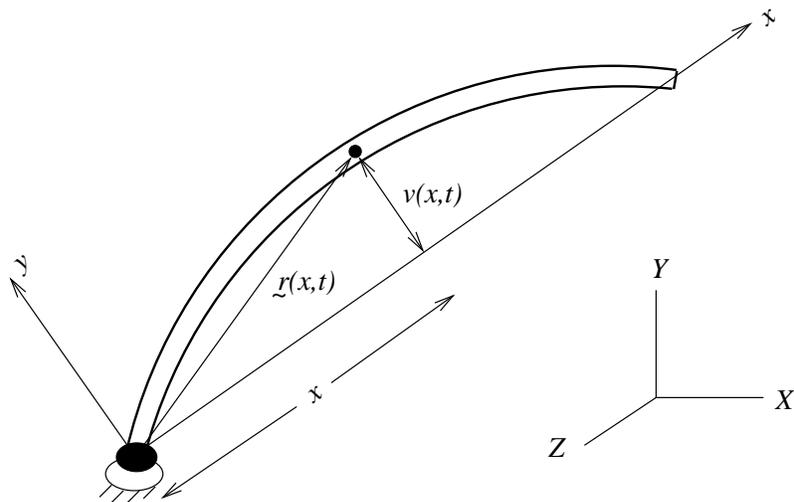
Figure 2. The flexible link.