Jourdain's variational equation and Appell's equation of motion for nonholonomic dynamical systems

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Based on Jourdain's variational equation proposed in 1909, we deduce a minimal set of general equations of motion for nonholomic dynamical systems of particles and rigid bodies. This equation of motion for the system, which differs slightly from the Gibbs–Appell equation, appears to be the same as the equation derived by Kane in 1961. Since the same equation was established by Appell in 1903 on the basis of D'Alembert's principle, the newly derived equation is named Appell's equation.
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I. INTRODUCTION

Analytical mechanics as embodied by Whittaker (1904)[1] has long been regarded as a mature subject. Therefore, the introduction of a new principle of mechanics or another equation of motion to rival Newton's equation would attract great attention. In a more recent treatise on the same subject by Pars (1965),[2] the author discussed a new fundamental form of equations in mechanics, Jourdain's variational equation, and a general set of equations of motion, the Gibbs–Appell equation. The latter was not new as it was discussed in the section entitled Appell's equation in Ref. 1. The former, which was postulated by Jourdain in 1909,[3] has generated considerable interest as discussed by Budo (1964),[4] Roberson and Schwertassek (1988),[5] Lesser (1995),[6] and Moon (1998).[7]

In 1961, another set of equations of motion was discussed by Kane[8] and was applied subsequently to dynamical problems with nonholonomic constraints, especially for multiconnected rigid bodies.[9–12] His work was later regarded by others[5] to be related to Jourdain's earlier work, and the proposed equation and its derivation were named Kane's method, Kane's equation, or Jourdain's method by different authors. However, Kane's method was considered to be equivalent to the Gibbs–Appell method by Desloge (1986),[13] and was noted by Chen (1984)[14,15] as an intermediate step in the derivation of the Gibbs–Appell equation from Gauss's principle.

In a series of papers, Jourdain[3,16,17] investigated the application of D'Alembert's principle, Hamilton's principle, and Gauss's principle to dynamical systems with nonholonomic constraints. He noted in Ref. 3 that in addition to the D'Alembert–Lagrange variational equation for the virtual displacement and that of Gauss–Gibbs for the virtual change of the acceleration, there could be another variational equation for the virtual change of velocity. He then postulated Jourdain's variational equation by inference from the other two equations. However, he gave no basic principles on which the variational equation was based, nor a new set of equations of motion. Nevertheless, his equation along with two other variational equations based on the aforementioned principles are listed as three of the six fundamental forms of mechanics.[2]

In this paper, we treat Jourdain's variational equation as a mathematical representation of a basic principle of mechanics, the principle of virtual power, which is on par with D'Alembert's principle and Gauss's principle. We then establish in Sec. III a minimal set of general equations of motion for nonholonomic dynamical systems from Jourdain's equation. Our original motivation was to find if the Gibbs–Appell equation or similar ones could also be derived from the new principle. However, there emerged a set of differential equations which is very much like Kane's equation of motion. Because Chen and Desloge already indicated its similarity to the Gibbs–Appell equation, we searched the literature and found that a similar intermediate differential equation also appeared in Appell's treatise (1903)[18] and Jourdain's paper (1905).[17] In fact, Appell's work was reproduced in the treatise by Whittaker in 1904,[1] containing the same equation as an intermediate step.

In Sec. IV, we show how Jourdain's variational equation can be applied to derive Euler's equation of linear momentum and of angular momentum for rigid bodies. An application of the variational equation and Appell's equation in generalized coordinates or quasi-coordinates is illustrated by a system of two rolling wheels mounted on a rigid axle. The paper concludes in Sec. V with comparisons of Appell's equation with Newton's equation, Lagrange's equation, and the Gibbs–Appell equation.

II. NEWTON'S PRINCIPLE AND OTHER PRINCIPLES OF MECHANICS

All principles of classical mechanics are founded on the basis of Newton's law for a single particle in free motion.[19] The law may be expressed in modern notation as

$$m\ddot{\mathbf{r}}=\mathbf{F}^{I}, \tag{1}$$

where m denotes the mass, $\mathbf{r}(t)$ is the position vector at time t of the particle from a fixed point of reference, and $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ are the velocity and the acceleration, respectively. The quantity $\mathbf{F}^{I}$ is the impressed (external) force acting on the particle, and is a known vector or vector function of t, $\mathbf{r}$, and $\dot{\mathbf{r}}$, but not of $\ddot{\mathbf{r}}$. (Note that the term "impressed force" is used in Newton's *Principia*.[19])

A. Newton's principle

If the motion of the particle is constrained, $\ddot{\mathbf{r}}$ might not be proportional to the impressed force. Let $\mathbf{F}=\mathbf{F}^I+\mathbf{F}^C$ be the resultant (total) force, where $\mathbf{F}^C$ denotes the force exerted by the constraints on the particle. The equation of motion is modified to be

$$m\ddot{\mathbf{r}}=\mathbf{F}=\mathbf{F}^I+\mathbf{F}^C. \tag{2}$$

In accordance with Newton's third law, $\mathbf{F}^C$ represents the reaction from the constraints. Equation (2) should be supplemented by auxiliary conditions for the unknown constraint force.

For a system of N interacting particles, we need to write an equation of motion for each particle and the conditions of constraint become more complicated as N increases. We identify the position of each particle by the subscript j ($j=1,2,\ldots,N$) in the equations of motion,

$$m_j\ddot{\mathbf{r}}_j=\mathbf{F}_j+\sum_{k=1(k\neq j)}^{N}\mathbf{f}_{jk}=\mathbf{F}_j^I+\mathbf{F}_j^C \quad (j=1,\ldots,N), \tag{3}$$

where $\mathbf{F}_j$ denotes the resultant force acting on each particle m_j by external agents, and $\mathbf{f}_{jk}$ denotes the interactive force between the pair of particles m_j and m_k. The force $\mathbf{f}_{jk}$ may be an impressed force or a constraint force depending on the interaction mechanism, and the sum is a combination of both as shown in the last part of Eq. (3). For example, if two particles are connected by a rigid and weightless rod, the constraints on m_j and m_k are represented by the unknown forces $\mathbf{f}_{jk}$ and $\mathbf{f}_{kj}=-\mathbf{f}_{jk}$, the equation of motion being supplemented by the geometrical constraint $|\mathbf{r}_{jk}|=l_0=\text{constant}$, where $\mathbf{r}_{jk}=\mathbf{r}_j-\mathbf{r}_k$. On the other hand, if the same particles are connected by an elastic spring with stiffness κ, the effect of the constraint is represented by $\mathbf{f}_{jk}=-\kappa(|\mathbf{r}_{jk}|-l_0)\mathbf{r}_{jk}/|\mathbf{r}_{jk}|$, which is considered to be an impressed force $\mathbf{F}_j^I$. These considerations might sound ambiguous and confusing, but it was the situation at the turn of eighteenth century. In fact, Newton did not discuss the constraint force, and the problem of a compound pendulum composed of a rigid rod and several attached masses was a much studied subject at that time. It was against this background that the search for an alternative to Newton's principle began shortly after the publication of the *Principia*.[19]

B. D'Alembert's principle

In 1743, D'Alembert proposed a principle of dynamics for the motion of a system of interconnected particles, that circumvented the difficulty of constraint forces. By combining his principle with the principle of virtual displacement proposed earlier by J. Bernoulli (1727), Lagrange (1788) established the following variational equation for a dynamical system with constraints,[20]

$$\sum_j (m_j\ddot{\mathbf{r}}_j-\mathbf{F}_j^I)\cdot\delta\mathbf{r}_j=0 \quad \text{(D'Alembert–Lagrange equation)}. \tag{4}$$

In Eq. (4) and the following, the summation over j is from 1 to N and the symbol δ means an arbitrary and infinitesimal change as defined in the calculus of variation. The virtual displacement $\delta\mathbf{r}_j$ is an arbitrary, infinitesimal, and instantaneous change of position vector $\mathbf{r}_j$ that is imposed on particle j with $\delta t=0$. In addition, the principle stipulates that the virtual displacement must be compatible with the constraints of the system, and the N values of $\delta\mathbf{r}_j$ are related by a set of constraint equations. Therefore, the condition of compatibility with the constraint is regarded as a part of the principle, not that of the definition of the virtual displacement.[21]

The conditions of constraint are usually prescribed in the form of functions g_s that relate the position coordinates or functions ϕ_s that relate the positions and velocities of the particles,

$$\text{geometric:} \quad g_s(\mathbf{r},t)=0 \quad (s=1,2,\ldots,K), \tag{5}$$

$$\text{kinematic:} \quad \phi_s(\mathbf{r},\dot{\mathbf{r}},t)=0 \quad (s=1,2,\ldots,L), \tag{6}$$

where $\mathbf{r}$ represents $(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$ and $\dot{\mathbf{r}}$ represents $(\dot{\mathbf{r}}_1,\dot{\mathbf{r}}_2,\ldots,\dot{\mathbf{r}}_N)$; the integers K and L are less than $3N$. A special type of ϕ_s, linear in $\dot{\mathbf{r}}_j$, which occurs frequently in the rotational motion of rigid bodies, is given below:

$$\sum_j \boldsymbol{\alpha}_j^s(\mathbf{r},t)\cdot\dot{\mathbf{r}}_j+a^s(\mathbf{r},t)=0 \quad (s=1,\ldots,L), \tag{7}$$

where $\boldsymbol{\alpha}_j^s$ represents a 3×1 column matrix with elements $\boldsymbol{\alpha}_{j1}^s$, $\boldsymbol{\alpha}_{j2}^s$, $\boldsymbol{\alpha}_{j3}^s$, and a^s is a scalar function.

In differential form, Eq. (7) can be written as

$$\sum_j \boldsymbol{\alpha}_j^s\cdot d\mathbf{r}_j+a^s\,dt=0. \tag{8}$$

Because the virtual displacement is infinitesimal and instantaneous, we retain the coefficients of $d\mathbf{r}_j$ and drop that of dt in Eq. (8) to obtain the conditions on $\delta\mathbf{r}_j$ as

$$\sum_j \boldsymbol{\alpha}_j^s\cdot\delta\mathbf{r}_j=0 \quad (s=1,\ldots,L). \tag{9}$$

If the differential form on the left-hand side of Eq. (8) is integrable so that the constraints (7) can be integrated to yield a form of geometric constraint, $g_s(\mathbf{r},t)=c_s$, where c_s is the integration constant, then the kinematic constraint condition (7) is called holonomic. Otherwise, it is called nonholonomic.

For a system of N particles with K geometrical constraints, the degrees of freedom in three dimensions is reduced from $3N$ to $n=3N-K$. Lagrange introduced n generalized coordinates $q_1,q_2,\ldots,q_n$ such that

$$\mathbf{r}_j=\hat{\mathbf{r}}_j(\mathbf{q},t), \tag{10}$$

and derived his celebrated equations of motion,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right)-\frac{\partial T}{\partial q_k}-Q_k=0 \quad (k=1,\ldots,n), \tag{11}$$

where $T=\frac{1}{2}\Sigma m_j\dot{\mathbf{r}}_j\cdot\dot{\mathbf{r}}_j$ is the total kinetic energy of the system, and $Q_k=\Sigma_j\mathbf{F}_j^I\cdot(\partial\mathbf{r}_j/\partial q_k)$ are the generalized forces. Equation (4), known as the D'Alembert–Lagrange variational equation, and Eq. (11) form the foundation of analytical dynamics.

For a system with nonholonomic constraints, Lagrange introduced the method of unspecified multipliers (Lagrange's multipliers) to reduce the variational equation to a system of individual differential equations involving $\ddot{\mathbf{r}}_j$ or $\ddot{q}_k$ and the unknown multipliers. However, the end result, which provides a systematic treatment of the constraint forces and the constraint conditions, is not much simpler than Newton's equations. The search for an alternative continued.[22]

C. Gauss's principle

The D'Alembert–Lagrange method is based on the combination of two principles, D'Alembert's principle and the principle of virtual displacement. There was a strong desire to have all of analytical mechanics founded on a single supposition. The principle of least constraint enunciated by Gauss in 1829 was the first successful attempt to base analytical mechanics on a single postulate, see, for example, Ref. 22.

The effect of constraints on the motion of a system can be represented by the constraint function $\mathcal{C}$, defined as $\mathcal{C} = \Sigma_j \frac{1}{2} m_j (\ddot{\mathbf{r}}_j - \mathbf{F}_j^I/m_j) \cdot (\ddot{\mathbf{r}}_j - \mathbf{F}_j^I/m_j)$. The condition of least constraint is then determined by minimizing $\mathcal{C}$, which is realized by setting the derivative or the variation of $\mathcal{C}$ equal to zero. We then obtain the following variational equation of motion;

$$\sum_j (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j^I) \cdot \delta_2 \ddot{\mathbf{r}}_j = 0 \quad \text{(Gauss–Gibbs variational equation)}, \tag{12}$$

where the virtual change of acceleration $\delta_2 \ddot{\mathbf{r}}_j$ must be compatible with the constraints, and $\delta_2 t = 0$, $\delta_2 \mathbf{r}_j = \delta_2 \dot{\mathbf{r}}_j = \mathbf{0}$.

The notation $\delta_2 \ddot{\mathbf{r}}_j$ in Eq. (12) and the idea of taking the variation of the acceleration while keeping the velocity and position unchanged were introduced by Gibbs,[23] where the index 2 indicates the second-order change of the position in dt. Actually, Gibbs postulated Eq. (12) by inference from the D'Alembert–Lagrange variational equation, and introduced the function

$$\mathcal{G} = \sum_j \frac{1}{2} m_j \ddot{\mathbf{r}}_j \cdot \ddot{\mathbf{r}}_j, \tag{13}$$

such that the variational equation (12) can be reduced to the following invariant form:

$$\delta_2 \mathcal{G} = \delta_2 Q, \tag{14}$$

where $\delta_2 \mathcal{G} = \Sigma_j (\partial \mathcal{G} / \partial \ddot{\mathbf{r}}_j) \cdot \delta_2 \ddot{\mathbf{r}}_j$ and $\delta_2 Q = \Sigma_j \mathbf{F}_j^I \cdot \delta_2 \ddot{\mathbf{r}}_j$. This formulation is in agreement with the principle of least constraint.

The variational equation (12), which we call the Gauss–Gibbs variational equation, can be applied to dynamical systems with nonholonomic constraints. For the system with L linear kinematical constraints, we can find a set of $l = 3N - L$ independent parameters, y_k $(k = 1,2,\ldots,l)$, from the conditions of constraint, so that all $3N$ components of $\ddot{\mathbf{r}}_j$ are related to $\ddot{y}_k$. The variation of $\ddot{\mathbf{r}}_j$ is then related to the variations of $\ddot{y}_k$ as

$$\delta_2 \ddot{\mathbf{r}}_j = \sum_{k=1}^{l} \frac{\partial \ddot{\mathbf{r}}_j}{\partial \ddot{y}_k} \delta_2 \ddot{y}_k. \tag{15}$$

Due to the independency of $\delta_2 \ddot{y}_k$, the variational equation (14) is satisfied if and only if

$$\sum_j \frac{\partial \mathcal{G}}{\partial \ddot{\mathbf{r}}_j} \cdot \frac{\partial \ddot{\mathbf{r}}_j}{\partial \ddot{y}_k} - \sum_j \mathbf{F}_j^I \cdot \frac{\partial \ddot{\mathbf{r}}_j}{\partial \ddot{y}_k} = 0 \quad (k = 1,2,\ldots,l). \tag{16}$$

If we replace the first term by the partial derivatives of the Gibbs function, we find

$$\frac{\partial \mathcal{G}}{\partial \ddot{y}_k} = \sum_j \mathbf{F}_j^I \cdot \frac{\partial \ddot{\mathbf{r}}_j}{\partial \ddot{y}_k} \quad (k = 1,2,\ldots,l), \tag{17}$$

where the function $\mathcal{G}(\ddot{\mathbf{r}})$ has been transformed to the function $\mathcal{G}(\ddot{y}_1, \ddot{y}_2, \ldots, \ddot{y}_l)$. The l equations of motion (17), plus the L conditions for the nonholonomic constraints, form the determined set of differential equations for the $3N$ components of $\mathbf{r}_j$, some of them having being transformed to y_k.

The equation of motion in the form of Eq. (17), which is derived from the intermediate form (16), was first established by Appell[24] based on D'Alembert's principle, and both equations were cited by Whittaker in Sec. 107 of his treatise.[1] Because the invariant form (14) was established earlier by Gibbs, the function $\mathcal{G}$ is known as the Gibbs function, and Eq. (17) is known as the Gibbs–Appell equation, whose importance was stressed by Pars (Ref. 2, p. 202) as "...provide what is probably the simplest and most comprehensive form of the equations of motion so far discovered. They are of superlatively simple form, they apply with equal facility to holonomic and to nonholonomic systems alike, and quasi-coordinates may be used freely."

D. Remarks

By the end of the nineteenth century, several other fundamental principles of mechanics had been developed, but are not discussed here. It could be said that the search for an alternative to Newton's principle for dynamical systems with constraints had reached a satisfactory state. There were D'Alembert's principle and Lagrange's equations for system with holonomic constraints, and Gauss's principle and the Gibbs–Appell equations for a system with holonomic or nonholonomic constraints. This satisfaction was changed by the launching of artificial satellites in 1957. Classical dynamics was ushered into the space age, and orbital mechanics, the dynamics of multiconnected rigid bodies, and many other subjects suddenly became active topics of research (see Refs. 25 and 26 and other books previously mentioned). A new round of search for alternatives emerged.

III. PRINCIPLE OF VIRTUAL POWER AND APPELL'S EQUATION OF MOTION

A. Jourdain's variational equation

The new search for an alternative actually began with the paper by Jourdain.[3] Following Gibbs,[23] he showed that the virtual velocity $\delta_1 \dot{\mathbf{r}}_j$, which varies the velocity $\dot{\mathbf{r}}_j$ with both position and time being fixed, satisfies the same conditions for linear kinematic constraints (7) as does the other two variations, $\delta \mathbf{r}_j$ and $\delta_2 \ddot{\mathbf{r}}_j$. By comparing all three variations, Jourdain established, again by inference, the following variational equation:

$$\sum_j (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j^I) \cdot \delta_1 \dot{\mathbf{r}}_j = 0 \quad \text{(Jourdain's variational equation)} \tag{18}$$

for $\delta_1 \dot{\mathbf{r}}_j$ ($\delta_1 t=0$, $\delta_1 \mathbf{r}_j=0$) compatible with the linear kinematic constraints, that is,

$$\sum_j \boldsymbol{\alpha}_j^s \cdot \delta_1 \dot{\mathbf{r}}_j = 0. \tag{19}$$

Equation (18) was considered to be intermediate in character between Eqs. (4) and (12).

Other than showing the derivation of the extension of Lagrange's equations to nonholonomic systems[1] directly from the variational equation (18), Jourdain did not give any physical meaning to Eq. (18), or a basic principle upon which the equation was founded. Nevertheless, Eq. (18), along with Eqs. (4) and (12), are regarded as the fundamental equations of mechanics by Pars.

Attempts have been made to derive Jourdain's equation (18) from the D'Alembert–Lagrange equation by first differentiating Eq. (4) with respect to time, showing the interchangeability of the operators (d/dt and δ), and then invoking the conditions $\delta \mathbf{r}_j = \mathbf{0}$.[27,28] In accordance with the classical definition of virtual displacement as adopted here, however, the quantity $\delta \mathbf{r}_j$ in Eq. (4) is the imposed arbitrary change of the position vector at a given point, not a change of some quantity along an arbitrary time-dependent path. One cannot take the time-derivative of a quantity that is not a function of time. Therefore, Eq. (18) should be treated as an independent variational equation of motion in mechanics. We call it *Jourdain's variational equation*, and regard it as a mathematical representation of the principle of virtual power because the product of force with virtual velocity is virtual power.[7] Based on this principle we shall establish a set of equations of motion for a nonholonomic system in Sec. III B and the Euler's equation of motion for rigid bodies in Sec. IV.

B. Appell's equations of motion

Consider again a system of N interconnected particles subject to L linear kinematic constraints in the form of Eq. (7), with K of the constraints being holonomic in the form of Eq. (5). By differentiating $\mathbf{r}_j(t)$ in Eq. (10) with respect to time, we obtain

$$\dot{\mathbf{r}}_j = \sum_{r=1}^{n} \frac{\partial \hat{\mathbf{r}}_j(\mathbf{q},t)}{\partial q_r} \dot{q}_r + \frac{\partial \hat{\mathbf{r}}_j(\mathbf{q},t)}{\partial t}, \tag{20}$$

where $\dot{q}_r = dq_r/dt$ are the *generalized velocities*. The acceleration is given by

$$\ddot{\mathbf{r}}_j = \sum_{r=1}^{n} \frac{\partial \hat{\mathbf{r}}_j(\mathbf{q},t)}{\partial q_r} \ddot{q}_r + \sum_{r,s=1}^{n} \frac{\partial^2 \hat{\mathbf{r}}_j(\mathbf{q},t)}{\partial q_r \partial q_s} \dot{q}_r \dot{q}_s + 2\sum_{r=1}^{n} \frac{\partial^2 \hat{\mathbf{r}}_j(\mathbf{q},t)}{\partial q_r \partial t} \dot{q}_r + \frac{\partial^2 \hat{\mathbf{r}}_j(\mathbf{q},t)}{\partial t^2}. \tag{21}$$

From Eqs. (20) and (21), we obtain the following relations for the coefficients of transformation:

$$\frac{\partial \mathbf{r}_j}{\partial q_r} = \frac{\partial \hat{\mathbf{r}}_j}{\partial q_r} = \frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{q}_r} = \frac{\partial \ddot{\mathbf{r}}_j}{\partial \ddot{q}_r}. \tag{22}$$

The system is said to have $l = 3N - L$ degrees of freedom represented by n generalized coordinates. If we take the δ_1 variation of Eq. (20), we find

$$\delta_1 \dot{\mathbf{r}}_j = \sum_{r=1}^{n} \frac{\partial \hat{\mathbf{r}}_j(\mathbf{q},t)}{\partial q_r} \delta_1 \dot{q}_r. \tag{23}$$

We then substitute Eqs. (21) and (23) into Eq. (18) and obtain the variational equation in generalized coordinates as

$$\sum_{r=1}^{n} \sum_j (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j^I) \cdot \frac{\partial \hat{\mathbf{r}}_j}{\partial q_r} \delta_1 \dot{q}_r = 0. \tag{24}$$

If all $\delta_1 \dot{q}_r$ are independent, we would recover Lagrange's equation immediately, as shown in Sec. V.

Because the generalized coordinates are still constrained by the remaining $m = L - K$ nonholonomic conditions, only l of them are independent. If we substitute Eq. (20) into the remaining nonholonomic part of Eq. (7), we obtain the following m equations relating the n generalized velocities:

$$\sum_{r=1}^{n} D_{sr}(\mathbf{q},t)\dot{q}_r + d_s(\mathbf{q},t) = 0 \quad (s=1,\ldots,m), \tag{25}$$

where the coefficients D_{sr} and d_s are given by

$$D_{sr} = \sum_j \boldsymbol{\alpha}_j^s \cdot \frac{\partial \hat{\mathbf{r}}_j}{\partial q_r}, \quad d_s = \sum_j \boldsymbol{\alpha}_j^s \cdot \frac{\partial \hat{\mathbf{r}}_j}{\partial t} + a^s. \tag{26}$$

In the previous set of constraint conditions (25) on $\dot{\mathbf{q}}$, only l generalized velocities are independent. We may select the independent ones, and relabel them as $\dot{y}_k$ ($k=1,\ldots,l$), which are called the *privileged velocities*.[2] We denote the remaining dependent generalized velocities by $\dot{z}_h$ ($h=1,\ldots,m$), and partition the matrix $[D_{sr}]$ in Eq. (25) into two parts, $[D'_{sh}]$ and $[D''_{sk}]$, such that

$$\sum_{h=1}^{m} D'_{sh}\dot{z}_h = -\sum_{k=1}^{l} D''_{sk}\dot{y}_k - d_s \quad (s=1,\ldots,m). \tag{27}$$

If we assume that $[D_{sr}]$ is of full rank, the matrix $[D'_{sh}]$ is nonsingular. Hence, we can express $\dot{z}_h$ in terms of $\dot{y}_k$ by solving Eq. (27) to obtain

$$\dot{z}_h = \sum_{k=1}^{l} B_{hk}\dot{y}_k + b_h \quad (h=1,\ldots,m), \tag{28}$$

where

$$B_{hk} = -\sum_{s=1}^{L} [D']_{hs}^{-1} D''_{sk}, \tag{29a}$$

$$b_h = -\sum_{s=1}^{L} [D']_{hs}^{-1} d_s \quad (h=1,\ldots,m) \quad (k=1,\ldots,n). \tag{29b}$$

Let $[P_{rs}]$ denote the permutation matrix from $(\dot{y}_k, \dot{z}_h)$ to $\dot{q}_r$, so that $\dot{q}_r = \Sigma_{k=1}^{l} P_{rk}\dot{y}_k + \Sigma_{h=1}^{m} P_{r(l+h)}\dot{z}_h$. Every generalized velocity can be then expressed in terms of the privileged velocities $\dot{y}_k$ as

$$\dot{q}_r = \sum_{k=1}^{l} C_{rk}\dot{y}_k + c_r, \tag{30}$$

where

$$C_{rk} = P_{rk} + \sum_{h=1}^{m} P_{r(l+h)} B_{hk}, \tag{31a}$$

$$c_r = \sum_{h=1}^{m} P_{r(l+h)} b_h . \tag{31b}$$

If we substitute the δ_1 variation of Eq. (30), that is,

$$\delta_1 \dot{q}_r = \sum_{k=1}^{l} C_{rk} \delta_1 \dot{y}_k \tag{32}$$

into the variational equation in generalized coordinates (24), we obtain

$$\sum_{k=1}^{l} \sum_{r=1}^{n} \sum_{j=1}^{N} (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j^I) \cdot \frac{\partial \hat{\mathbf{r}}_j}{\partial q_r} C_{rk} \delta_1 \dot{y}_k = 0. \tag{33}$$

Because the $\delta_1 \dot{y}_k$ are arbitrary and independent, the following equations of motion are established:

$$\sum_{r=1}^{n} \sum_{j=1}^{N} (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j^I) \cdot \frac{\partial \hat{\mathbf{r}}_j}{\partial q_r} C_{rk} = 0 \quad (k = 1,\ldots,l). \tag{34}$$

On the other hand, we can bypass the using of virtual generalized velocities $\delta_1 \dot{q}_r$ and work with $\delta_1 \dot{\mathbf{r}}_j$ directly. If Eq. (30) is substituted into Eq. (20), the following transformation from $\dot{y}_k$ to $\dot{\mathbf{r}}_j$ is obtained:

$$\dot{\mathbf{r}}_j = \sum_{k=1}^{l} \boldsymbol{\beta}_j^k (\mathbf{z},\mathbf{y},t) \dot{y}_k + \boldsymbol{\beta}_j (\mathbf{z},\mathbf{y},t), \tag{35}$$

where

$$\boldsymbol{\beta}_j^k = \sum_{r=1}^{n} \frac{\partial \hat{\mathbf{r}}_j}{\partial q_r} C_{rk}, \quad \boldsymbol{\beta}_j = \sum_{r=1}^{n} \frac{\partial \hat{\mathbf{r}}_j}{\partial q_r} c_r + \frac{\partial \hat{\mathbf{r}}_j}{\partial t}. \tag{36}$$

Like $\boldsymbol{\alpha}_j^s$, the $\boldsymbol{\beta}_j^k$ is an assemblage of the coefficients of the transformation in vector notation.

By taking the time-derivative of Eq. (35), the accelerations $\ddot{\mathbf{r}}_j$ are found in terms of the privileged coordinates as

$$\ddot{\mathbf{r}}_j = \sum_{k=1}^{l} \boldsymbol{\beta}_j^k \ddot{y}_k + \dot{\boldsymbol{\beta}}_j^k \dot{y}_k + \dot{\boldsymbol{\beta}}_j . \tag{37}$$

From Eqs. (35) and (37), we note that

$$\frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{y}_k} = \frac{\partial \ddot{\mathbf{r}}_j}{\partial \ddot{y}_k} = \boldsymbol{\beta}_j^k , \tag{38}$$

which holds even for linear nonholonomic kinematic constraints.

According to Eq. (35), the virtual velocity $\delta_1 \dot{\mathbf{r}}_j$ is related to $\delta_1 \dot{y}_k$ as

$$\delta_1 \dot{\mathbf{r}}_j = \sum_{k=1}^{l} \boldsymbol{\beta}_j^k \delta_1 \dot{y}_k . \tag{39}$$

If we substitute the transformation of $\delta_1 \dot{\mathbf{r}}_j$ into Jourdain's variational equation (18) and transform the dependent variables in $\ddot{\mathbf{r}}_j$ and $\mathbf{F}_j^I$ accordingly, we obtain

$$\sum_{j=1}^{N} \sum_{k=1}^{l} (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j^I) \cdot \boldsymbol{\beta}_j^k \delta_1 \dot{y}_k = 0. \tag{40}$$

We recognize that the $\delta_1 \dot{y}_k$ are independent and deduce the following set of independent equations of motion in privileged coordinates:

$$\sum_j m_j \ddot{\mathbf{r}}_j (\mathbf{z},\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},t) \cdot \boldsymbol{\beta}_j^k = \sum_j \mathbf{F}_j^I (\mathbf{z},\mathbf{y},\dot{\mathbf{y}},t) \cdot \boldsymbol{\beta}_j^k \quad (k = 1,\ldots,l). \tag{41}$$

These equations may also be derived directly from Eq. (34) by noting the relation (36); they form a system of second-order differential equations in $\ddot{y}_k$, the coefficients are functions of y_k, $\dot{y}_k$ and the dependent variables. This system of $l = 3N - L$ equations plus the m supplementary kinematic equations of constraints (25) form a determinant set of differential equations for the n unknown variables q_r. We shall name Eq. (41) as well as Eq. (34) Appell's equations of motion for the reason discussed in Sec. VI.

C. Reduced Appell's equations of motion and cyclic coordinates

The final form of Appell's equations (41) together with the supplemental conditions varies according to the choice of the privileged velocities. This set of equations can sometimes be reduced to a fewer number of equations by choosing the privileged velocities judiciously.

Note that Eq. (41) is independent of the velocities $\dot{z}_h$, but is still dependent on z_h. In addition, Eq. (28) may be used as the supplementary conditions for the constraints. Accordingly, we may reduce the number of supplemental equations by one if one of the dependent variables, say z_c, is absent in B_{hk}, b_h, $\boldsymbol{\beta}_j^k$, $\boldsymbol{\beta}_j$, and the forcing functions $\mathbf{F}_j^I$, because the corresponding constraint equation for $\dot{z}_c$ in Eq. (28) is redundant. The total number of equations to be solved is thus reduced from n to $n-1$. Because such a reduction is similar to the notion of cyclic coordinates for the Lagrangian,[29] we call the variable z_c the *cyclic* coordinate, and the reduced system of the equations the *reduced Appell's equations*. If all the dependent coordinates are cyclic, we have

$$\sum_j m_j \ddot{\mathbf{r}}_j (\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},t) \cdot \boldsymbol{\beta}_j^k (\mathbf{y},t) = \sum_j \mathbf{F}_j^I (\mathbf{y},\dot{\mathbf{y}},t) \cdot \boldsymbol{\beta}_j^k (\mathbf{y},t) \quad (k = 1,\ldots,l) \tag{42}$$

with the minimal number (l) of independent differential equations for the reduced system, which is precisely the degrees of freedom of the constrained system. As observed by Appell,[18] this reduced minimal set of equations is sufficient to determine all privileged coordinates y_k, without invoking the equation of constraints. Once the variables y_k are solved, the remaining variables z_h are found by quadrature through Eq. (28).

IV. VARIATIONAL EQUATION OF MOTION FOR SYSTEM OF RIGID BODIES

In Sec. III, the variational equations and the equations of motion were given for discrete systems with a finite number of particles. The motion of rigid bodies, however, is usually treated by applying Euler's laws for rigid body motion. Since a rigid body may be considered as an aggregate of an infinite number of particles or continuously distributed mass elements separated by constant distances, the variational equation for discrete systems may be extended to that of rigid bodies by replacing the summation by the integration over the mass elements. Various procedures for deducing the equation of motion for a rigid body from the variational equation have been proposed (see, for example, Refs. 30–32). The one proposed by Wittenburg[31] does not invoke *a priori* knowledge of Euler's law and is summarized below.

A. Equations based on D'Alembert's principle

For a continuous distribution of mass in a rigid body $\mathcal{B}$ occupying a volume in three-dimensional space, a particle with mass m_j at the position $\mathbf{r}_j$ is replaced by the mass element dm occupying the position $\mathbf{r}$ which varies continuously within the volume. The total mass of the body is given by $m=\int_{\mathcal{B}} dm$, and the center of mass of the body is defined as $\mathbf{r}^c=\int_{\mathcal{B}} \mathbf{r}\, dm/m$. If $d\mathbf{F}^I$ denotes the force impressed on the mass element dm, the D'Alembert–Lagrange equation (4) may be extended for a continuum as

$$\int_{\mathcal{B}} (\ddot{\mathbf{r}}\, dm - d\mathbf{F}^I)\cdot\delta\mathbf{r} = \delta W\,, \tag{43}$$

where δW is the virtual work done by the internal forces. Because the mutual distance between each and every pair of mass elements is held fixed for a rigid body, we set $\delta W=0$ in Eq. (43).

The general motion of a rigid body may be decomposed into the motion of a reference point in the body plus a rotation about that point. We choose the reference point to be the center of mass, and denote the relative position vector of constant length from the center of mass C to each mass element by $\mathbf{r}'$. Due to its infinitesimal character, the virtual displacement of each mass element positioned at $\mathbf{r}$ can be expressed as

$$\delta\mathbf{r} = \delta\mathbf{r}^c + \delta\boldsymbol{\theta}\times\mathbf{r}'\,, \tag{44}$$

where $\delta\boldsymbol{\theta}$ contains three infinitesimal angles of rotation. We denote the angular velocity of the rigid body by $\boldsymbol{\omega}$, and write

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}^c + \boldsymbol{\omega}\times\mathbf{r}'\,, \tag{45a}$$

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}^c + \dot{\boldsymbol{\omega}}\times\mathbf{r}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')\,. \tag{45b}$$

If we substitute Eq. (45) into Eq. (43) and carry out the integration over the body $\mathcal{B}$, we obtain the variational equation of motion for a rigid body:

$$(m\ddot{\mathbf{r}}^c - \mathbf{F}^I)\cdot\delta\mathbf{r}^c + (\mathbf{I}^c\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathbf{I}^c\cdot\boldsymbol{\omega} - \mathbf{L}^c)\cdot\delta\boldsymbol{\theta} = 0, \tag{46}$$

where $\mathbf{F}^I=\int_{\mathcal{B}} d\mathbf{F}^I$ is the total impressed force, $\mathbf{L}^c=\int_{\mathcal{B}}\mathbf{r}' \times d\mathbf{F}^I$ is the total impressed torque on the entire body about the point C, and $\mathbf{I}^c$ denotes the moment of inertia dyadic about the center of mass,

$$\mathbf{I}^c = \int_{\mathcal{B}} (|\mathbf{r}'|^2\mathbf{1} - \mathbf{r}'\mathbf{r}')\, dm. \tag{47}$$

If the rigid body is free to translate or rotate, $\delta\mathbf{r}^c$ and $\delta\boldsymbol{\theta}$ are independent of each other. The variational equation (46) leads to two independent equations of motion, one from each term,

$$m\ddot{\mathbf{r}}^c = \mathbf{F}^I, \quad \mathbf{I}^c\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathbf{I}^c\cdot\boldsymbol{\omega} = \mathbf{L}^c. \tag{48}$$

They are, respectively, the well-known Euler's equation of linear momentum and that of angular momentum for a rigid body in free motion.

We note that Euler's equation of angular momentum can also be derived from Lagrange's equations in terms of the quasi-velocities $\boldsymbol{\omega}$, which are linear functions of the rate of change of the orientation angles (see Ref. 1, Sec. 30). The notion of quasi-velocities was also adopted by Poincaré to derive the same equation directly from Hamilton's principle by considering a special variation of the angular velocity.[33] In modern geometric mechanics, Poincaré's work has been incorporated with the theory of Lie groups to form the Euler–Poincaré equations applicable to rigid bodies and ideal fluids (Ref. 34, Chap. 13).

B. Equations based on principle of virtual power

The derivation of the variational equation for a rigid body based on D'Alembert's principle as outlined in Sec. IV A is essentially that of Wittenburg.[31] Because the motion of rigid bodies is usually constrained by kinematic conditions, we convert the virtual displacements to virtual velocities. By reasoning similar to that used by Jourdain,[3] we replace $\delta\mathbf{r}$ by $\delta_1\dot{\mathbf{r}} = \delta_1\dot{\mathbf{r}}^c + \delta_1\boldsymbol{\omega}\times\mathbf{r}'$ in Eq. (43) and obtain the following variational equation for a rigid body:

$$(m\ddot{\mathbf{r}}^c - \mathbf{F}^I)\cdot\delta_1\dot{\mathbf{r}}^c + (\mathbf{I}^c\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathbf{I}^c\cdot\boldsymbol{\omega} - \mathbf{L}^c)\cdot\delta_1\boldsymbol{\omega} = 0. \tag{49}$$

Again Euler's equations for a rigid body in free motion Eq. (48) are recovered.

For a system of N_B rigid bodies, we append a subscript $j=1,\ldots,N_B$ to all physical variables pertaining to each body, and sum all the terms for $j=1,\ldots,N_B$ to obtain a single variational equation for the entire system,

$$\sum_{j=1}^{N_B} (m_j\ddot{\mathbf{r}}_j^c - \mathbf{F}_j^I)\cdot\delta_1\dot{\mathbf{r}}_j^c + (\mathbf{I}_j^c\cdot\dot{\boldsymbol{\omega}}_j + \boldsymbol{\omega}_j\times\mathbf{I}_j^c\cdot\boldsymbol{\omega}_j - \mathbf{L}_j^c)\cdot\delta_1\boldsymbol{\omega}_j = 0. \tag{50}$$

This single variational equation can be reduced to a system of independent equations of motion for multiconnected rigid bodies if the kinematic constraint conditions for $\dot{\mathbf{r}}_j^c$ and $\boldsymbol{\omega}_j$ are given.

From the constraint conditions, a set of generalized coordinates $q_1, q_2, \ldots, q_n$ may be constructed such that

$$\dot{\mathbf{r}}_j^c = \sum_{r=1}^{n} \frac{\partial\hat{\mathbf{r}}_j^c(\mathbf{q},t)}{\partial q_r}\dot{q}_r + \frac{\partial\hat{\mathbf{r}}_j^c(\mathbf{q},t)}{\partial t}, \tag{51a}$$

$$\boldsymbol{\omega}_j = \sum_{r=1}^{n} \hat{\boldsymbol{\omega}}_{jr}(\mathbf{q},t)\dot{q}_r\,. \tag{51b}$$

These generalized coordinates may be subject to the other nonholonomic constraints in the form of Eq. (25). By a process similar to that presented in Sec. III B, a set of l independent privileged velocities $\dot{y}_k$ $(k=1,\ldots,l)$ may be found, in terms of which the generalized velocities are expressed as in Eq. (30). The following set of equations of motion analogous to Eq. (34) is then established:

$$\sum_{r=1}^{n}\left[\sum_{j=1}^{N_B} (m_j\ddot{\mathbf{r}}_j^c - \mathbf{F}_j^I)\cdot\frac{\partial\hat{\mathbf{r}}_j^c}{\partial q_r} + (\mathbf{I}_j^c\cdot\dot{\boldsymbol{\omega}}_j + \boldsymbol{\omega}_j\times\mathbf{I}_j^c\cdot\boldsymbol{\omega}_j - \mathbf{L}_j^c)\cdot\hat{\boldsymbol{\omega}}_{jr}\right]C_{rk} = 0. \tag{52}$$

On the other hand, from Eqs. (51) and (30), we find

$$\dot{\mathbf{r}}_j^c = \sum_{k=1}^{l} \overline{\boldsymbol{\beta}}_j^k\dot{y}_k + \overline{\boldsymbol{\beta}}_j\,, \tag{53a}$$

$$\boldsymbol{\omega}_j = \sum_{k=1}^{l} \overline{\boldsymbol{\gamma}}_j^k\dot{y}_k + \overline{\boldsymbol{\gamma}}_j\,, \tag{53b}$$

where

$$\overline{\boldsymbol{\beta}}_j^k=\sum_{r=1}^{n}\frac{\partial \hat{\mathbf{r}}_j^c}{\partial q_r}C_{rk}, \quad \overline{\boldsymbol{\beta}}_j=\sum_{r=1}^{n}\frac{\partial \hat{\mathbf{r}}_j^c}{\partial q_r}c_r+\frac{\partial \hat{\mathbf{r}}_j^c}{\partial t}, \tag{54a}$$

$$\overline{\boldsymbol{\gamma}}_j^k=\sum_{r=1}^{n}\hat{\boldsymbol{\omega}}_{jr}C_{rk}, \quad \overline{\boldsymbol{\gamma}}_j=\sum_{r=1}^{n}\hat{\boldsymbol{\omega}}_{jr}c_r. \tag{54b}$$

The δ_1 variations of the velocities are then related by

$$\delta_1\dot{\mathbf{r}}_j^c=\sum_{k=1}^{l}\overline{\boldsymbol{\beta}}_j^k\delta_1\dot{y}_k, \tag{55a}$$

$$\delta_1\boldsymbol{\omega}_j=\sum_{k=1}^{l}\overline{\boldsymbol{\gamma}}_j^k\delta_1\dot{y}_k. \tag{55b}$$

From either Eq. (50) or Eq. (52), another form of the equations of motion for rigid bodies can be established as

$$\sum_j (m_j\ddot{\mathbf{r}}_j^c-\mathbf{F}_j^I)\cdot\overline{\boldsymbol{\beta}}_j^k+(\mathbf{I}_j^c\cdot\dot{\boldsymbol{\omega}}_j+\boldsymbol{\omega}_j\times\mathbf{I}_j^c\cdot\boldsymbol{\omega}_j-\mathbf{L}_j^c)\cdot\overline{\boldsymbol{\gamma}}_j^k$$
$$=0 \quad (k=1,\ldots,l). \tag{56}$$

Once the coefficient vectors $\overline{\boldsymbol{\beta}}_j^k$ and $\overline{\boldsymbol{\gamma}}_j^k$ are found from the constraint conditions, the minimal set of equations of motion for a system of rigid bodies is established directly.

To summarize, we have shown three approaches to analyzing the motion of a system of rigid bodies based on the variational equation of virtual power. The first is to apply directly the variational equation for a rigid body, Eq. (50), together with the conditions of kinematic constraints. The second is to apply the equation of motion in generalized coordinates, Eq. (52), and the third is to use the equation of motion in quasi-velocities $\boldsymbol{\omega}_j$, Eq. (56). There is no simple guideline for which approach one should follow. Customarily, the use of the quasi-velocities $\boldsymbol{\omega}_j$ as independent variables is preferred for torque-free motion ($\mathbf{L}_j^c=0$). Otherwise, one usually replaces $\boldsymbol{\omega}_j$ by the appropriate generalized velocities such as the Eulerian angular speeds or the angular rate of roll–pitch–yaw angles. These three approaches are illustrated with the example given in Sec. IV C.

C. The motion of a two-wheel-axle assemblage

To illustrate the application of the variational equation of virtual power for dynamical systems with kinematic constraints, we consider the motion of two wheels of radius a mounted on a rigid axle of length l, rolling on a rough plane inclined by an angle α to the horizontal (see Fig. 1). Each wheel is attached by a bearing to the axle, free to spin about the axle with no wobbling about any axis perpendicular to the axle.

We associate the coordinate axes x, y, z with a triad of unit vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to the inclined plane; the x and z axes are along and normal to the inclined plane, respectively. The motion of each wheel is then specified by six parameters, three Cartesian coordinates (x_j, y_j, z_j) $(j=1,2)$ of the center of mass and three Eulerian angles $(\phi_j, \theta_j, \psi_j)$ for each wheel in free motion, rotating about a set of axes through the center of each wheel as shown in Fig. 1. The triad is first rotated about $\mathbf{k}$ through the precession angle ϕ_j to the new position indicated by the triad $\{\mathbf{i}_j', \mathbf{j}_j', \mathbf{k}_j'=\mathbf{k}\}$ such that $\mathbf{j}_j'$ is along the axle. The axle is then rotated through the nutation angle θ_j about $\mathbf{i}_j'$ to the new triad $\{\mathbf{i}_j''=\mathbf{i}_j', \mathbf{j}_j'', \mathbf{k}_j''\}$ such that $\mathbf{j}_j''$ coincides

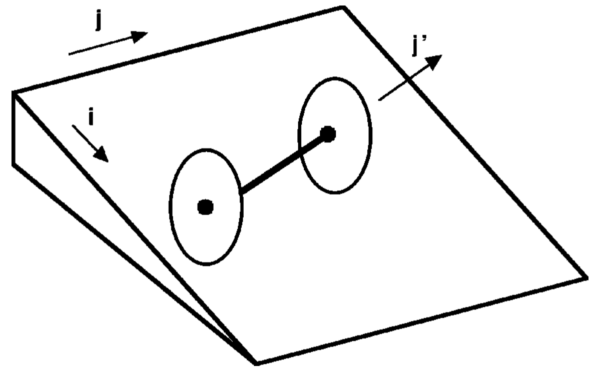

Fig. 1. Wheel-axle assemblage moving on an inclined plane with inclination angle α. In addition to the axes shown, the $\mathbf{k}$ axis points normal to the plane. The angles of rotation of the wheels (not shown) are ψ_1 and ψ_2, and the heading of the assemblage is represented by the angle ϕ between $\mathbf{j}$ and $\mathbf{j}'$.

with the axis of symmetry of the wheel j. Finally, the axes are rotated through the spin angle ψ_j around $\mathbf{j}_j''$. The resultant angular velocity for each wheel is then

$$\boldsymbol{\omega}_j=\dot{\phi}_j\mathbf{k}+\dot{\theta}_j\mathbf{i}_j'+\dot{\psi}_j\mathbf{j}_j'', \tag{57}$$

and hence

$$\delta_1\boldsymbol{\omega}_j=(\delta_1\dot{\phi}_j)\mathbf{k}+(\delta_1\dot{\theta}_j)\mathbf{i}_j'+(\delta_1\dot{\psi}_j)\mathbf{j}_j''. \tag{58}$$

Let M be the total mass for the wheel assemblage and the load, concentrated at the center of each wheel, and let m' ($<M$) be the mass of the rim of the wheel. The impressed force acting on each wheel is thus $Mg\sin\alpha\mathbf{i}+Mg\cos\alpha\mathbf{k}$, and the impressed torque about the center of mass is zero. The principal moment of inertia dyadic for each wheel is given by

$$\mathbf{I}_j^c=\tfrac{1}{2}m'a^2\mathbf{i}_j''\mathbf{i}_j''+m'a^2\mathbf{j}_j''\mathbf{j}_j''+\tfrac{1}{2}m'a^2\mathbf{k}_j''\mathbf{k}_j''. \tag{59}$$

Because the wheels remain in contact with the ground, and the axle remains parallel to the $x-y$ plane, we have two simple conditions of constraints for each wheel,

(i) $\dot{z}_1=0$ (or $z_1=a$),

(ii) $\dot{\theta}_1=0$ (or $\theta_1=0$),

(iii) $\dot{z}_2=0$ (or $z_2=a$),

(iv) $\dot{\theta}_2=0$ (or $\theta_2=0$).

By setting the nutation angles $\theta_1=\theta_2=0$, the triad $\{\mathbf{i}_j'', \mathbf{j}_j'', \mathbf{k}_j''\}$ coincides with $\{\mathbf{i}_j', \mathbf{j}_j', \mathbf{k}_j'\}$. In addition, both wheels precess at the same angle with the axle, giving rise to the geometrical condition $\phi_1=\phi_2\equiv\phi$, or the kinematic condition

(v) $\dot{\phi}_1=\dot{\phi}_2=\dot{\phi}$,

and $\{\mathbf{i}_1', \mathbf{j}_1', \mathbf{k}_1'\}=\{\mathbf{i}_2', \mathbf{j}_2', \mathbf{k}_2'\}\equiv\{\mathbf{i}', \mathbf{j}', \mathbf{k}'\}$.

The fact that each wheel rolls without sliding at the contact point imposes two nonholonomic kinematic conditions of constraints on each wheel:

(vi) $\dot{x}_1=a\dot{\psi}_1\cos\phi$,

(vii) $\dot{y}_1=a\dot{\psi}_1\sin\phi$,

(viii) $\dot{x}_2=a\dot{\psi}_2\cos\phi$,

(ix) $\dot{y}_2 = a\dot{\psi}_2 \sin\phi$.

The assumption that the axle is rigid and inextensible imposes a geometrical constraint condition, $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$, which may be converted into a kinematic condition, as either

$$\dot{x}_2 = \dot{x}_1 - l\dot{\phi}\cos\phi, \tag{60a}$$

or

$$\dot{y}_2 = \dot{y}_1 - l\dot{\phi}\sin\phi. \tag{60b}$$

If we combine Eq. (60a) with conditions (vi) and (viii) to eliminate $\dot{x}_1$ and $\dot{x}_2$, we find

$$\text{(x)} \quad a\dot{\psi}_2 = a\dot{\psi}_1 - l\dot{\phi}, \tag{61}$$

which can also be derived from Eq. (60b) with conditions (vii) and (ix). Therefore, there are altogether ten equations of kinematic constraints, six being integrable and four being nonintegrable. The number of degrees of freedom of the system of two connected rigid bodies is reduced from 12 to 2 by the constraints from the ground and the axle. Note that among the ten constraint conditions, (i)–(iv) and (vi)–(ix) are the constraints on individual wheels, while (v) and (x) are the constraints on the two coupled rigid bodies.

After the constraint conditions have been specified, any of the three approaches discussed in Sec. IV B can be adopted to derive the minimal set of equations of motion as shown below.

Method I (direct application of the variational equation): From the first five conditions of constraints, we find $\delta_1\dot{z}_1 = \delta_1\dot{\theta}_1 = \delta_1\dot{z}_2 = \delta_1\dot{\theta}_2 = 0$, and $\delta_1\dot{\phi}_1 = \delta_1\dot{\phi}_2 \equiv \delta_1\dot{\phi}$. The variational equation of virtual power (50) for the assemblage is then reduced to

$$\sum_{j=1}^{2} \left((M\ddot{x}_j - Mg\sin\alpha)\delta_1\dot{x}_j + M\ddot{y}_j\delta_1\dot{y}_j + m' a^2\ddot{\psi}_j\delta_1\dot{\psi}_j \right) + m' a^2\ddot{\phi}\delta_1\dot{\phi} = 0. \tag{62}$$

By choosing $\dot{\psi}_1$ and $\dot{\phi}$ as the privileged velocities, the remaining five constraints are used to find the following relations:

$$\delta_1\dot{x}_1 = a\cos\phi\,\delta_1\dot{\psi}_1, \quad \delta_1\dot{y}_1 = a\sin\phi\,\delta_1\dot{\psi}_1, \tag{63a}$$

$$\begin{aligned} \delta_1\dot{x}_2 &= -l\cos\phi\,\delta_1\dot{\phi} + a\cos\phi\,\delta_1\dot{\psi}_1, \\ \delta_1\dot{y}_2 &= -l\sin\phi\,\delta_1\dot{\phi} + a\sin\phi\,\delta_1\dot{\psi}_1, \end{aligned} \tag{63b}$$

$$\delta_1\dot{\psi}_2 = -\frac{l}{a}\delta_1\dot{\phi} + \delta_1\dot{\psi}_1. \tag{63c}$$

The linear and angular accelerations in Eq. (62) are derived by differentiating the respective constraint conditions with respect to time. If we substitute all these expressions into Eq. (62) and collect coefficients for $\delta_1\dot{\psi}_1$ and $\delta_1\dot{\phi}$, we obtain two independent equations of motion for the wheel assemblage,

$$2a(M+m')\ddot{\psi}_1 - l(M+m')\ddot{\phi} = 2Mg\sin\alpha\cos\phi, \tag{64a}$$

$$al(M+m')\ddot{\psi}_1 - (Ml^2 + m'l^2 + m'a^2)\ddot{\phi} = Mgl\sin\alpha\cos\phi. \tag{64b}$$

Note that only two variables ϕ and ψ_1 and their time derivatives appear in Eq. (64), which can be solved without any supplementary conditions of constraints. Once this system of second-order differential equations is solved for ϕ and ψ_1 with the appropriate initial conditions, the remaining five velocities are determined from the constraint conditions and the corresponding coordinates are determined by simple quadrature.

Method II (application of Appell's equation in generalized coordinates): After eliminating the five holonomic constraints, a set of seven generalized coordinates $[q_r] \equiv (x_1, y_1, \psi_1, x_2, y_2, \psi_2, \phi)$ is constructed. The remaining five conditions of constraints may be expressed in the form of Eq. (25) as

$$\sum_{r=1}^{7} D_{sr}\dot{q}_r = 0 \quad (s = 1,\ldots,5), \tag{65}$$

where the elements of the first row in the 5×7 coefficient-matrix $[D_{sr}]$ are $D_{1r}(1,0,-a\cos\phi,0,0,0,0)$; the second to the last (fifth) row is, respectively, $D_{2r}(0,1,-a\sin\phi,0,0,0,0)$, $D_{3r}(0,0,0,1,0,-a\cos\phi,0)$, $D_{4r}(0,0,0,1,0,-a\sin\phi,0)$, $D_{5r}(0,0,-a,0,0,a,l)$. Because the variable ϕ appears in the coefficient-matrix $[D_{sr}]$, we select $\dot{\phi}$ as the first privileged velocity $\dot{Y}_1$ according to the guideline outlined in Sec. III. In addition, we select $\dot{\psi}_1$ as the second privileged velocity $\dot{Y}_2$ and solve Eq. (65) to obtain the transformation

$$\dot{q}_r = \sum_{k=1}^{2} C_{rk}\dot{Y}_k, \tag{66}$$

where the 7×2 transformation matrix $[C_{rk}]$ is given by two columns, the transpose of the first column: $C_{r1}(0,0,0,-l\cos\phi,-l\sin\phi,-l/a,1)$; the second column is $C_{r2}(a\cos\phi,a\sin\phi,1,a\cos\phi,a\sin\phi,1,0)$. Because all five dependent variables x_1, y_1, x_2, y_2, ψ_2 are absent from the matrix $[C_{rk}]$ and the impressed force $\mathbf{F}^I$, they constitute the five cyclic coordinates for the system. If we substitute the coefficient matrix $[C_{rk}]$ into Appell's equation in generalized coordinates, Eq. (52), the same set of equations as in Eq. (64) can be established. They constitute what we called the reduced Appell's equations, after ignoring the five cyclic coordinates x_1, y_1, x_2, y_2, ψ_2.

Method III (application of Appell's equation with quasi-velocities): After identifying the privileged velocities $\dot{\psi}_1$ and $\dot{\phi}$, the coefficient vectors in Eq. (53) for the velocities are found to be

$$\overline{\boldsymbol{\beta}}_1^{\psi_1} = \overline{\boldsymbol{\beta}}_2^{\psi_1} = a\mathbf{i}', \quad \overline{\boldsymbol{\gamma}}_1^{\psi_1} = \overline{\boldsymbol{\gamma}}_2^{\psi_1} = \mathbf{j}'', \tag{67a}$$

$$\overline{\boldsymbol{\beta}}_1^{\phi} = \mathbf{0}, \quad \overline{\boldsymbol{\beta}}_2^{\phi} = -l\mathbf{i}', \quad \overline{\boldsymbol{\gamma}}_1^{\phi} = \mathbf{k}, \quad \overline{\boldsymbol{\gamma}}_2^{\phi} = \mathbf{k} - \frac{l}{a}\mathbf{j}''. \tag{67b}$$

If we substitute these vectors, the accelerations, and the forces into Appell's equation with quasi-velocities (56), we again derive the same set of equations (64). This approach is essentially the same as the one used by Kane in establishing the equations of motion.[8]

V. COMPARISONS WITH OTHER PRINCIPLES AND EQUATIONS OF MOTION

A. Newton's principle and constraint forces

The reduction of the D'Alembert–Lagrange variational equation to Newton's equation of motion (3) is well known. It is established by applying Lagrange's method of unspecified multipliers. Analogously, we can reduce Jourdain's equation (18) by the same method. If we multiply each of the equations of constraint (19) by the unspecified multiplier λ_i and combine all constraint equations with the variational equation (18), we obtain

$$\sum_j \left(m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j^I - \sum_{s=1}^{L} \lambda_s \boldsymbol{\alpha}_j^s \right) \cdot \delta_1 \dot{\mathbf{r}}_j = 0. \tag{68}$$

By applying Lagrange's method, this single equation gives rise to the following system of independent equations of motion:

$$m_j \ddot{\mathbf{r}}_j = \mathbf{F}_j^I + \sum_{s=1}^{L} \lambda_s \boldsymbol{\alpha}_j^s \quad (j=1,\dots,N). \tag{69}$$

We thus recover Newton's principle from Jourdain's equation if the summation of $\lambda_s \boldsymbol{\alpha}_j^s$ is identified as the constraint force $\mathbf{F}_j^C$ in Eq. (3), that is,

$$\mathbf{F}_j^C = \sum_{s=1}^{L} \lambda_s \boldsymbol{\alpha}_j^s \quad (j=1,\dots,N). \tag{70}$$

Conversely, by multiplying each of Newton's equations (3) by $\delta_1 \dot{\mathbf{r}}_j$, and combining all the equations, we find

$$\sum_j m_j \ddot{\mathbf{r}}_j \cdot \delta_1 \dot{\mathbf{r}}_j - \sum_j \mathbf{F}_j^I \cdot \delta_1 \dot{\mathbf{r}}_j = \sum_j \mathbf{F}_j^C \cdot \delta_1 \dot{\mathbf{r}}_j . \tag{71}$$

If we use the expression of $\mathbf{F}_j^C$ in Eq. (70), which could also be established by D'Alembert's principle, the term on the right-hand side of Eq. (71), which represents the total virtual power generated by the constraint forces, can be expressed as

$$\sum_{j=1}^{N} \mathbf{F}_j^C \cdot \delta_1 \dot{\mathbf{r}}_j = \sum_{j=1}^{N} \sum_{s=1}^{L} \sum_{k=1}^{l} \lambda_s \boldsymbol{\alpha}_j^s \cdot \boldsymbol{\beta}_j^k \delta_1 \dot{y}_k . \tag{72}$$

The triple summation is seen to vanish on account of the conditions

$$\sum_j \boldsymbol{\alpha}_j^s \cdot \boldsymbol{\beta}_j^k = 0 \quad (s=1,\dots,L) \quad (k=1,\dots,l), \tag{73}$$

which can be derived from Eqs. (39) and (19). Hence, we recover Jourdain's variational equation from Newton's principle. Either way we have made use of the additional condition (70) which is supplemented by another principle. Therefore, Jourdain's variational equation and Newton's principle should be treated as independent equations in mechanics.

The deduction of Eq. (72) reveals that the total virtual power generated by the constraint forces vanishes if the virtual velocities are compatible with the prescribed kinematic constraints. This was mentioned as an independent principle in mechanics known as Jourdain's principle.[4] If we accept this principle, we can then directly derive Jourdain's variational equation from Newton's equation; this approach is called Jourdain's method.[5]

Furthermore, if we multiply each of Newton's equations (3) by the vector $\boldsymbol{\beta}_j^k$ directly, and then add these equations together, the grand sum is a single equation for the independent parameter y_k:

$$\sum_j m_j \ddot{\mathbf{r}}_j \cdot \boldsymbol{\beta}_j^k = \sum_j (\mathbf{F}_j^I + \mathbf{F}_j^C) \cdot \boldsymbol{\beta}_j^k . \tag{74}$$

By replacing $\mathbf{F}_j^C$ in Eq. (74) by the expression (70), we obtain

$$\sum_j m_j \ddot{\mathbf{r}}_j \cdot \boldsymbol{\beta}_j^k - \sum_j \mathbf{F}_j^I \cdot \boldsymbol{\beta}_j^k = \sum_j \sum_{s=1}^{L} \lambda_s \boldsymbol{\alpha}_j^s \cdot \boldsymbol{\beta}_j^k . \tag{75}$$

Again the double sum on the right-hand side of Eq. (75) vanishes. Equation (75) is then identical to Appell's equation (41).

This derivation clarifies the meaning of $\mathbf{F}_j^C$, $\mathbf{F}_j^I$ in Eq. (3) and subsequent equations. All interactive forces, including the contact of a particle with its surroundings, which are unknown quantities but are accompanied by prescribed kinematic conditions of constraints, are treated as the constraint forces $\mathbf{F}_j^C$. On the other hand, those that are specified explicitly in terms of unknown kinetic variables and pertinent kinetic coefficients (material constants), such as the forces exerted by the connecting spring, are treated as the impressed forces $\mathbf{F}_j^I$.

Within the framework of modern geometrical mechanics,[35] the velocities $\dot{\mathbf{r}}_j$ reside in a fiber of the tangent bundle of the configuration space, and the kinematic conditions of constraints restrict the admissible velocity to be within a subspace of the fiber. Accordingly, the virtual velocity $\delta_1 \dot{\mathbf{r}}_j$ compatible with the constraints is an arbitrary vector in the subspace, and the expression in Eq. (39) shows that the admissible subspace is spanned by the base vectors $\boldsymbol{\beta}_j^k$ $(j=1,\dots,N)$. Hence the result $\Sigma_j \mathbf{F}_j^C \cdot \boldsymbol{\beta}_j^k = 0$ which leads to Eq. (41) from Eq. (75) can be viewed as the vanishing of the projection of the constraint forces onto the admissible subspace spanned by $\boldsymbol{\beta}_j^k$. Appell's equation then implies that the net forces $\mathbf{F}_j^I - m_j \ddot{\mathbf{r}}_j$ are orthogonal to the base vectors in the admissible subspace. Similarly, Jourdain's variational equation in the form of Eq. (18) means that the net forces are orthogonal to the entire admissible subspace. This interpretation is in agreement with the projection method of dynamics recently proposed.[36–38]

B. Lagrange's equation

If all constraints of the system are holonomic, the generalized coordinates $q_1, q_2, \dots, q_n$ are all (independent) privileged coordinates y_k. We thus obtain

$$\boldsymbol{\beta}_j^k = \frac{\partial \mathbf{r}_j}{\partial q_k} = \frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{q}_k}. \tag{76}$$

Appell's equation (41) is readily reducible to Lagrange's equation (11) by the substitution of $\boldsymbol{\beta}_j^k$ with $\partial \mathbf{r}_j / \partial q_k$. For nonholonomic systems the reduction to the Lagrange–Ferrers equation[1] and other similar ones is also possible.

C. Gibbs–Appell equation

It was mentioned previously[15] that Appell's equation (41) is an intermediate step in deriving the Gibbs–Appell equation (17). In terms of the formulation presented in Sec. III, the derivation of one from the other is immediate. Note that in all three variational equations, the vector $m_j \ddot{\mathbf{r}}_j$ may be replaced by $\partial \mathcal{G} / \partial \ddot{\mathbf{r}}_j$, where $\mathcal{G}$ is given in Eq. (13), and for the case of linear kinematic constraints, Eq. (38) holds. The two parts of Eq. (41) can thus be expressed as

$$\sum_j m_j \ddot{\mathbf{r}}_j \cdot \boldsymbol{\beta}_j^k = \sum_j m_j \ddot{\mathbf{r}}_j \cdot \frac{\partial \ddot{\mathbf{r}}_j}{\partial \ddot{y}_k} = \frac{\partial \mathcal{G}}{\partial \ddot{y}_k}, \tag{77a}$$

$$\sum_j \mathbf{F}_j^I \cdot \boldsymbol{\beta}_j^k = \sum_j \mathbf{F}_j^I \cdot \frac{\partial \ddot{\mathbf{r}}_j}{\partial \ddot{y}_k}, \tag{77b}$$

and Eq. (16) is recovered. The reverse procedure of deriving Eq. (41) from the Gibbs–Appell equation (17) is equally simple.

In the applications of both Eqs. (17) and (41), there is no difference between the final form of the differential equations of motion if the same privileged coordinates are chosen, see, for example, Ref. 13. Equation (41) is however preferred to Eq. (17) as the former does not require the evaluation of the function $\partial \mathcal{G}(\ddot{y}_k, \dot{y}_k, y_k, \dot{z}_h, z_h, t) / \partial \ddot{y}_k$, see, for example, Ref. 39. Therefore, whatever is said about the importance of the Gibbs–Appell equation as quoted in the remarks of Sec. II applies equally well to Appell's equation (41), especially the reduced Appell's equation (42).

VI. CONCLUDING REMARKS

From 1899 to 1903, Appell[18] discussed new forms of equations of motion based on D'Alembert's principle and the variational equation (4) for dynamical systems with nonholonomic constraints. He first showed for a system of l degrees of freedom that there exists independent parameters $y_1, \ldots, y_l$ (privileged coordinates in our notation) such that $\delta \mathbf{r}_j = \Sigma \boldsymbol{\beta}_j^k \delta y_k$, in our notation. The general equations of motion can then be written in compact form (Ref. 18, Eq. (5), p. 304) as follows:

$$Q_k - P_k = 0 \quad (k = 1, \ldots, l), \tag{78}$$

where $Q_k = \Sigma_j \mathbf{F}_j \cdot \boldsymbol{\beta}_j^k$ and $P_k = \Sigma_j m_j \ddot{\mathbf{r}}_j \cdot \boldsymbol{\beta}_j^k$. He then noted the selection of the independent parameters from the generalized coordinates for a nonholonomic system, and relation (38) was established (Ref. 18, Sec. 465). By writing $P_k = \partial \mathcal{G} / \partial \ddot{y}_k$, where $\mathcal{G}$ is the Gibbs' function, Appell then deduced Eq. (17), without assigning the name to the function $\mathcal{G}$ or the equation. The entire presentation including Eqs. (17) and (41) has been discussed succinctly by Whittaker (Ref. 1, Sec. 207).

However, the utilitarian value of Eq. (41) remained unnoticed until Kane,[8] who derived the same equations from "D'Alembert's principle" (Eq. (18) in Ref. 8), which is essentially an alternative form of Eq. (3) in this paper. By multiplying each equation in Eq. (3) by the vector $\boldsymbol{\beta}_j^k$, which was called the *partial velocity* corresponding to the *generalized speed* $\dot{y}_k$, and summing over all the particles, Kane established the equations of motion in the form of Eq. (74). Although the term of virtual displacement is not used, the notion of instantaneous and infinitesimal change is implied in stating the "equation of instantaneous constraints." He then deduced the final equation in the form of Eq. (41) by asserting without proof that $\Sigma_j \mathbf{F}_j^C \cdot \boldsymbol{\beta}_j^k$ vanishes. The proof is supplied in Sec. V C, Eqs. (70)–(74), by invoking Newton's principle and Jourdain's principle concurrently.

In this paper, Jourdain's variational equation is postulated as a mathematical representation of a fundamental principle in mechanics, the principle of virtual power, independent of all other principles. We then derived in Sec. III the fundamental equations of motion (41) from Jourdain's equation. Because the method we used and the equations we derived are not much different from those used by Appell, Eq. (41) is named Appell's equation.

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THE CYCLOTRON

In 1931, Lawrence and his co-workers succeeded in building the first cyclotron, using a tank six inches across and a small electromagnet whose poles faced each other vertically across the gap. In the gap was placed a shallow cylindrical tank, pumped out to a high vacuum so that the particles inside could move freely without interference from air molecules. Lawrence fed deuterons (heavy hydrogen nuclei) as atomic projectiles in at the center and kicked them around at high speeds using a radio frequency oscillator. He then graduated to a bigger setup, using a huge eighty-five-ton magnet and a vacuum tank eight inches across, which allowed him to accelerate the deuterons at very high speeds and direct them against any target. His work developing powerful beams of particles had already earned high praise from none other than Bohr himself, ‘‘the dean of quantum theorists,’’ who would make two trips from Copenhagen to California in the 1930’s to check up on the young Berkeley Physicist.

Jennett Conant, *Tuxedo Park* (Simon & Schuster, New York, NY, 2002), p. 134.